Institutional Investors, Heterogeneous Benchmarks and the Comovement of Asset Prices

ANDREA M. BUFFA
Boston University

IDAN HODOR
Hebrew University

March 2017*

Abstract

We study the equilibrium implications of an economy in which asset managers are each subject to a different benchmark. We demonstrate how heterogeneous benchmarking endogenously generates a mechanism through which fundamental shocks propagate across assets. Heterogeneous benchmarking reduces short-run return correlation, and may even lead to negative asset comovement. An asset that is included in a benchmark can not only become negatively correlated with assets included in a different benchmark, but also with assets belonging to the same benchmark. Our results are in line with the weakened comovement across industry-sector portfolios and investment styles over short horizons, and provide new testable implications on the established “asset-class” effect. The presence of institutional investors with different benchmarks also triggers additional price pressure amplifying return volatility. Our setting is tractable and we obtain our results in closed-form.

*Contacts: buffa@bu.edu and idan.hodor@mail.huji.ac.il. We thank Suleyman Basak, Ron Kaniel, Ellen Paulus, Anna Pavlova, Fernando Zapatero, seminar participants at Hebrew University, University of Colorado Boulder, Stockholm School Economics, University of Texas at Austin, and conference participants at Northeastern University Finance Conference for helpful comments and discussions.
1 Introduction

A characteristic feature of the professional money management industry is that the performance of asset managers is measured relative to the performance of a designated benchmark. Within mutual fund families, for example, every asset manager is assigned a specific benchmark in accordance with her investment style. Moreover, in a bid to compete for investors, investment funds increasingly diversify their mandates and investment styles (Kacperczyk, Sialm, and Zheng, 2005). Therefore, the benchmarks that are used to affect asset managers’ incentives have become increasingly heterogeneous. In this paper, we take the presence of heterogeneous benchmarking as given and study the implications of such heterogeneity on equilibrium asset prices, with a particular focus on their comovements. To our knowledge, ours is the first attempt to analyze this form of heterogeneity in an equilibrium setting.

We consider a dynamic general equilibrium model with two asset managers who, besides caring for their own performance, are each concerned with the performance of a different benchmark. We follow Basak and Pavlova (2013) and model benchmarking concerns by embedding them in the asset managers’ objective functions. Our economy has one riskless bond and multiple risky assets, among which some are part of a benchmark and some are not. The bond is in zero net supply, while the risky assets are in positive net supply. We model each manager’s benchmark as consisting of two risky assets: a common asset, that is part of both benchmarks, and a specialized asset, that is exclusive to a single benchmark. One interpretation of the specialized assets in our economy is that they represent different sector portfolios, or alternatively different investment styles, as in Barberis and Shleifer (2003). By varying the weights of the specialized assets relative to the common asset within each benchmark, our formulation allows us to vary the degree of benchmark heterogeneity in the economy.

A major advantage of our framework is that it preserves analytical tractability and allows us to characterize the equilibrium quantities in closed-form. Our main goal is to understand how cashflow news propagates across risky assets when asset managers have heterogeneous benchmarks. In order to isolate the equilibrium effects due to heterogeneous benchmarking, we intentionally ignore aggregate shocks that would affect all assets. We demonstrate that heterogeneous benchmarking can lead to contagion of cashflow news across assets belonging to different benchmarks. Accordingly, we characterize a rich structure of asset price comovements that delivers new testable implications.

We first examine the equilibrium asset return exposures to cashflow news. To this purpose, we develop a decomposition of these exposures in three parts: (i) a fundamental component; (ii) a component due to homogeneous benchmarking; and (iii) a component capturing the effects of heterogeneous benchmarking. The first two parts of this decomposition confirm the established results that cashflow
news for an asset that is included in a benchmark is amplified, and spills over to all the other assets within its benchmark. The third component of our decomposition isolates the effects of the mechanism propagating cashflow news that is solely attributable to heterogeneous benchmarking.

We find that in the presence of heterogeneous benchmarking, cashflow news affecting the fundamentals of a specialized asset spills over to the asset returns of a specialized asset in a different benchmark. To fix ideas, this result implies that news about a telecom-stock in a telecom-sector benchmark, affects the returns of a utility-stock in a utility-sector benchmark. Importantly, this spillover effect does not rely on the benchmarks having overlapping assets. In particular, we uncover that heterogeneous benchmarking causes the return of a specialized asset to become negatively exposed to cashflow news about other specialized assets.

The intuition behind these results goes as follows. Benchmarking incentives create hedging motives that induce managers to increase their demand for assets in their benchmark. The hedging demand of one manager increases the prices of the assets in her benchmark, and when other managers are subject to different benchmarks, they effectively perceive these assets as “overvalued”. By exploiting this overvaluation, which is induced by heterogeneous benchmarking, managers can further increase the exposure to the assets in their own benchmark. Therefore, as positive cashflow news on a specialized asset materializes, a manager who is benchmarked against that asset buys more of it, financing her purchase by selling off the other specialized asset she finds overvalued. The ensuing price pressures generate negative spillovers across the specialized assets of different benchmarks.

We further establish that heterogenous benchmarking is also responsible for additional spillover effects between specialized and common assets belonging to the same benchmark, over and above what homogenous benchmarking predicts. Following a positive shock to the specialized asset in her benchmark, a manager faces a new trade-off with respect to the common asset. If on one hand she wants to buy the common asset to be closer to her benchmark, on the other hand she also want to sell it to exploit the overvaluation induced by the other manager. We show that in equilibrium whether a manager ultimately buys or sells the common asset depends only on the weight of that asset in her benchmark, relative to the other manager’s benchmark. When this relative weight is higher, she buys the common asset because her benchmarking incentives dominate her desire to exploit its overvaluation. When it is lower, instead, she sells the common asset. Consequently, the direction of these additional spillovers induced by heterogeneous benchmarking can be positive or negative depending on the relative degree of specialization of the benchmarks in the economy. These findings provide a refinement to the established “asset-class” effect (e.g., Barberis, Shleifer and Wurgler, 2005; Boyer, 2011) that can be taken to the data.
Our key result is that asset return correlation can become negative in equilibrium. Notably, negative return correlation occurs not only for assets across different benchmarks, but also for assets within the same benchmark. This is a consequence of the negative asset return exposures, which are uniquely induced by heterogeneous benchmarking. Negative asset return comovements across benchmarks are in line with weakened comovements across asset styles that Barberis and Shleifer (2003) obtain within a behavioral model. We, instead, derive our results in a fully rational setting in general equilibrium, in which managers care about their performance relative to designated benchmarks, capturing their investment styles. We further show that assets within the same benchmark can also become negatively correlated if the benchmark is sufficiently specialized towards its style.

Our last set of results pertains asset return volatility. We find that heterogeneous benchmarking may alter the implicit link between benchmark weights and price pressures. In particular, we show that even when the total weight of a common asset across all benchmarks exceeds the weight of a specialized asset, in equilibrium, the common asset may be less volatile than the specialized one.

Equilibrium implications of explicit and implicit incentives in the asset management industry are the focus of a growing literature. Our paper relates most closely to the strand of that literature that focuses on benchmarking incentives. Brennan (1993) considers for the first time a static setting in which equilibrium prices are determined by the presence of institutional investors with preferences depending on performance relative to a benchmark. He shows that equilibrium expected returns are given by a two-factor model, with the factors being the market portfolio and the benchmark. Basak and Pavlova (2013) also embed benchmarking concerns into the objective function of a single institutional investor. They consider a dynamic setting, and allow for wealth effects to play a role in the determination of asset prices. In equilibrium, the institutional investor demands ore of asset included in her benchmark, thus raising the prices of these assets and making them more volatile and more correlated with each other.

Cuoco and Kaniel (2011) model explicit benchmarking incentives through delegation contracts, in a dynamic setting with two risky assets. A representative manager’s fee is a piece-wise affine function of absolute return and of the return relative to a benchmark. The authors show that while the manager’s demand always raises the prices of the assets included in the benchmark, the equilibrium effect on volatility depends on the convexity of the managers fee. Sotes-Paladino and Zapatero (2015) also consider convex incentives. They build on Basak, Pavlova, and Shapiro (2007), and introduce asymmetric information between a representative asset manager and a retail investor, giving rise to misvaluation from the perspective of the asset manager. While the price of the single risky asset is endogenous, it is by assumption determined only by the trading of the retail investor. In equilibrium, large overvaluation is associated with the manager outperforming the benchmark, which in turn induces her to engage in
indexing and forgo the opportunity to exploit the misvaluation. Buffa, Vayanos and Woolley (2014) study the joint determination of optimal asset management contracts and equilibrium prices. They show that benchmarking is part of an optimal contract in the presence of agency frictions, and can generate price implications that are in line with documented anomalies for the risk-return relationship and the pricing of the aggregate market.¹

A common feature among these theoretical works is the focus on a representative institution. We complement the existing analyses by considering a model with multiple risky assets and multiple institutions that are subject to different exogenous benchmarks. Notwithstanding the multiplicity of assets and institutions, our setting remains tractable and allows us to investigate how the cross-section of asset price comovements is affected by heterogeneous benchmarking. Our results on price comovements are most closely related to the literature on style investing.

Within a behavioral setting in which some investors use exogenously specified rules to move funds across investment styles, Barberis and Shleifer (2003) show that assets within the same style comove too much, whereas assets in different styles comove too little. Moreover, news about one investment style can affect the returns of assets in an unrelated style. Their model helps explain the asset-class effect that is documented in Barberis, Shleifer and Wurgler (2005), and Boyer (2011). Our model of heterogeneous benchmarking provides a rational framework that microfound the optimal demand for assets in different investment styles. Our results on negative spillovers are in line with the weakened comovements across investment styles, which other rational models of asset pricing, to our knowledge, are not able to generate without assuming an exogenous correlation structure of asset fundamentals. Ehling and Heyerdahl-Larsen (2016) study time variation in stock market correlations in a general equilibrium model in which agents have heterogeneous preferences due to external habits. While their model generates time varying correlation due to changes in the aggregate risk aversion, asset price comovements are always positive. Our model of heterogeneous benchmarking, which instead captures a structural feature of the asset management industry, is also able to deliver negative price comovements.²

The remainder of the paper is organized as follows. Section 2 presents the model. In Section 3 we characterize the equilibrium asset prices, and obtain implications for expected returns and for the


optimal risk exposure of asset managers. Section 4 presents our main results on asset price comovements. Section 5 concludes.

2 Model

In this section we lay out the structure of the model and introduce heterogeneous benchmarking. We consider a standard pure-exchange finite horizon economy, where time $t$ is continuous and goes from zero to $T$. The financial market is complete and risk is characterized by an $(N+1)$-dimensional Brownian motion $\omega = (\omega_1, ..., \omega_{N+1})'$, defined on the probability measure $\mathbb{P}$.

2.1 Assets

There are $N+1$ risky assets and a single riskless bond traded in the economy. The exogenous interest rate $r$ paid by the riskless bond is set to zero. Risky asset $k$ represents a claim on the terminal dividend $D_{kT}$, payed at time $T$, which is determined by the process

$$dD_{kt} = D_{kt} \left( \mu dt + \sigma_k D' d\omega_t \right),$$

for $k = 1, ..., N$, where $\mu$ is a positive constant and $\sigma_k D'$ is a vector with entry $k$ equal to $\sigma > 0$ and all other entries equal to zero, implying independent asset fundamentals. We refer to changes in $D_{kt}$ as cashflow news about asset $k$, and we assume that $D_{k0} > 0$. To maintain a tractable setting, we follow the approach in Basak and Pavlova (2013) and in Menzly, Santos, and Veronesi (2004), by directly modeling the dynamics of the aggregate dividend $D_T = \sum_{k=1}^{N+1} D_{kT}$ as

$$dD_t = D_t \left( \eta dt + \nu' d\omega_t \right),$$

where $\eta$ is a positive scalar and $\nu$ is a positive vector with equal entries equal to $\bar{\nu} > 0$. Each risky asset is in positive net supply of one share, and the price of asset $k$, denoted by $S_{kt}$, has the posited dynamics

$$dS_{kt} = S_{kt} \left( \mu_{kt} dt + \sigma_{kt} S' d\omega_t \right),$$

By leaving the dynamics of asset $(N+1)$’s cashflow news unspecified, this approach allows us to model the aggregate dividend as a geometric Brownian motion, which significantly enhances the tractability of our multi-asset economy. In an extension of this setting in which we instead model all $N+1$ asset dividends as geometric Brownian motions, we can no longer obtain analytical solutions but confirm our main findings through numerical methods.
for \( k = 1, \ldots, N + 1 \), where the price drift \( \mu^S_{kt} \) and its vector of volatilities \( \sigma^S_{kt} \) are endogenous quantities to be determined in equilibrium. We denote with \( dR_{kt} \equiv dS_{kt}/S_{kt} \) the return of asset \( k \) at time \( t \), and with \( dR^M_t \equiv \sum_k dR_{kt} \) the market return.

### 2.2 Asset Managers

Our economy is populated with two asset management firms, whose performance is evaluated against a different designated benchmark, reflecting different investment styles. That asset managers care about their performance relative to a specific benchmark index is a salient feature of the professional money management industry. This can be due to the implicit incentives induced by fund flows (e.g., Chevalier and Ellison, 1997; Sirri and Tufano, 1998), or by the explicit bonus incentives which depend on the relative performance of the fund (e.g., Ma, Tang, and Gomez, 2016). A second important feature is that asset managers care more to beat their benchmark when the benchmark is high than when it is low.

We model these key features in reduced form following Basak and Pavlova (2013), and specify that each asset manager \( i \)'s preferences are characterized by an increasing marginal utility in the level of its own benchmark. Formally, we adopt the following objective function, defined over the terminal wealth \( W_{iT} \),

\[
    u_i(W_{iT}) = \left(1 + b_i S^L_i \right) \log(W_{iT}),
\]

where \( b_i > 0 \), and \( S^L_i \) represents the terminal value of manager \( i \)'s benchmark.\(^4\) Given the dynamics in (3), asset manager \( i \)'s wealth evolves according to the following process

\[
    dW_{it} = W_{it} \pi^t_i \left( \mu^S_i dt + \Sigma^S_i d\omega_i \right),
\]

where \( \pi^t_i \) is a vector of portfolio allocations invested in each asset, \( \mu^S_i \) is a vector of price drifts \( \mu^S_{kt} \), and \( \Sigma^S_i \) is a matrix where the \( k^{th} \) row is asset \( k \)'s vector of volatilities \( \sigma^S_{kt} \). Asset manager \( i \) is endowed at time 0 with \( \lambda_i \) shares of the total asset market, such that \( \sum_{i=1}^2 \lambda_i = 1 \).

\(^4\)Given the presence of both explicit and implicit benchmarking incentives, there is no consensus on how these incentives exactly shape the objective function of assets managers. Koijen (2014) represents a first effort to structurally estimate the form of an asset manager’s objective function. A study that adopts a similar formulation as Basak and Pavlova (2013), to capture status-based incentives, is Hong, Jiang, Wang and Zhao (2014).
2.3 Benchmarking

The different investment styles of asset managers are reflected in the composition of their benchmarks. In particular, despite sharing some common assets, what makes benchmarks heterogeneous is that they also include specialized assets that are not included in other benchmarks. For example, the specialized assets of one benchmark may be growth stocks, while for another benchmark these may be value stocks, or alternatively, benchmarks may be specialized in different industries.

In what follows, we take benchmark heterogeneity as given, and characterize the assets in our economy by their inclusion in the two benchmarks. Accordingly, we define four asset classes out of the first \(N\) assets, each one corresponding to a different combination of benchmark inclusion: assets that are only included in the benchmark of manager 1; assets that are only included in the benchmark of manager 2; assets that are included in the benchmarks of both managers; and assets that are in neither benchmark. Since within each of these asset classes, every asset would exhibit the same equilibrium properties, we consider a representative asset of for each of the four asset classes (\(N = 4\)). Therefore, in the remainder of the paper, asset 1 denotes the specialized asset in benchmark 1, asset 2 denotes the specialized asset in benchmark 2, asset 3 denotes the common asset across both benchmarks, and asset 4 denotes the asset that is not benchmarked by either manager.

A further layer of heterogeneity may arise from the degree of specialization, which is the relative weight a benchmark places on its specialized assets relative to the common assets. Accordingly, we define the terminal value of each benchmark \(I_i^T\) in our economy as a geometric weighted average of its specialized asset \(i\) (for \(i = 1, 2\)) and the common asset 3, with weights given by \(\alpha_i\), and \(1 - \alpha_i\), respectively:\(^5\)

\[
I_{1T} = D_{1T}^{\alpha_1} D_{3T}^{1 - \alpha_1}, \quad I_{2T} = D_{2T}^{\alpha_2} D_{3T}^{1 - \alpha_2}.
\]  

In what follows, the vectors \(\chi_1\) and \(\chi_2\) denote the benchmarks’ weights in all the risky assets in the economy: \(\chi_1 = (\alpha_1, 0, 1 - \alpha_1, 0, 0)'\) and \(\chi_2 = (0, \alpha_2, 1 - \alpha_2, 0, 0)'\). A simple application of Itô’s lemma gives us the dynamics of cashflow news associated with benchmark \(i\),

\[
dI_{it} = I_{it} \left( \mu_i^t dt + \sigma_i^t d\omega_i \right),
\]

for \(i = 1, 2\), where \(\mu_i^t = \mu - \sigma^2 \alpha_i (1 - \alpha_i)\), and \(\sigma_i^t = \sigma \chi_i\).

\(^5\)We focus our analysis on geometric averages instead of arithmetic averages for the construction of benchmarks, as it preserves the tractability of the setting without altering the economic mechanism at play.
3 Equilibrium with Heterogeneous Benchmarking

In this section we present the equilibrium asset prices, the ensuing expected returns and managers’ exposures to the shocks in the economy. The illustration of the equilibrium in this section lays the ground for the discussion of asset price comovements in the next section. The equilibrium concept in our economy is standard and is defined as follows.

Definition 1 (Equilibrium). Asset prices and portfolio processes \( \{ S_{kt} \}_{k=1}^{N+1} \) and \( \{ \pi_i^t \}_{i=1}^2 \) form an equilibrium if: (i) given \( \{ S_{kt} \}_{k=1}^{N+1} \), \( \pi_i^t \) maximizes the expected value of (4) subject to the dynamic budget constraint in (5), for \( i = 1, 2 \); (ii) given \( \{ \pi_i^t \}_{i=1}^2 \), markets clear: \( \sum_{i=1}^2 W_i^T = D_T \).

The next theorem characterizes the equilibrium asset prices in closed form.

Theorem 1 (Asset Prices). The price-dividend ratios at time \( t \) of the assets included in the benchmarks are equal to

\[
\frac{S_{it}}{D_{it}} = \frac{S_{4t}}{D_{4t}} \left( 1 + \kappa_{it} f_t(\alpha_i) \right), \quad \frac{S_{3t}}{D_{3t}} = \frac{S_{4t}}{D_{4t}} \left( 1 + \sum_{i=1}^2 \kappa_{it} f_t(1 - \alpha_i) \right),
\]

for \( i = 1, 2 \), where \( \frac{S_{4t}}{D_{4t}} = e^{(\mu - \nu' \sigma_i^D)(T-t)} \) is the price-dividend ratio of the asset not included in the benchmarks, \( f_t(x) = e^{x \sigma^2 (T-t)} - 1 \) is a deterministic and monotonic transformation of \( x \), and

\[
\kappa_{it} = \lambda_i \frac{b_i I_{it} e^{(\mu_i - \nu' \sigma_i^D)(T-t)}}{1 + b_i I_{0i} e^{\mu_i T}} \left( \sum_{j=1}^2 \frac{\lambda_j}{1 + b_j I_{0j} e^{\mu_j T}} \left( 1 + b_j I_{jt} e^{(\mu_j - \nu' \sigma_j^D)(T-t)} \right) \right)^{-1}.
\]

Consequently, in equilibrium an asset included in a benchmark exhibits a higher price-dividend ratio than an asset not included in any benchmark. Moreover, the market price of risk at time \( t \) is equal to the vector

\[
\theta_t = \nu - \sigma \kappa_{1t} \chi_1 - \sigma \kappa_{2t} \chi_2.
\]

Theorem 1 explicitly characterizes the equilibrium asset prices in an economy in which asset managers are subject to heterogeneous benchmarking. In line with recent results in the literature (e.g., Cuoco and Kaniel, 2011; Basak and Pavlova, 2013; Buffa, Vayanos and Woolley, 2014; Sotes-Paladino and Zapatero, 2015), we find that benchmarking increases the prices of assets that are included in a benchmark, as
compared to assets that are not. The intuition underlying this result is that benchmarking concerns prompt asset managers to increase their demand for the assets that are included in their designated benchmark. To offset this increased demand, equilibrium prices for benchmarked assets go up, thus enabling market clearing. Accordingly, Theorem 1 also reveals that the equilibrium market price of risk is affected by the two benchmarks in the economy. This can be seen from (10), in which $\kappa_{it}$ represents the sensitivity of the market price of risk to the portfolio of benchmark $i$, $\chi_i$. The inclusion of an asset in one or both benchmarks reduces the price of risk of that asset, hence lowering its equilibrium Sharpe ratio. The intuition underlying the reduced price of risk is that the managers “like” to load on their benchmarks due to their incentives, and therefore, in equilibrium, “demand” less compensation for the risk associated with the benchmarked assets.

The state-variable $\kappa_{it}$ is an important equilibrium quantity, and can be thought of as the market price of risk associated with benchmark $i$. A key feature of this state-variable is that it depends positively on cashflow news to benchmark $i$, and negatively on cashflow news to benchmark $j$. The dependence of $\kappa_{it}$ on $D_{jt}$ (through $I_{jt} = D_{jt}^{\rho_j} D_{jt}^{1-\rho_j}$), plays an important role in the propagation of shocks across assets, which we explore in detail in the next section.

Given the equilibrium market prices of risk $\theta_t$, we can represent the (instantaneous) expected return for risky assets with a three factor representation. The first factor is the return of the market. The second factor is the return on the first benchmark. And the third factor is the return on the second benchmark. We formalize this representation in the following proposition.

**Proposition 1 (Expected Returns).** Expected asset returns at time $t$ take the following three-factor representation:

$$
\mathbb{E}_t(dR_t) = \Sigma_t^{S'}(\Sigma_t^{S'}\Sigma_t^{S})^{-1}\left[\nu_t\text{Cov}_t(dR_t, dR^M_t) - \sigma_1\kappa_{it}\text{Cov}_t(dR_t, dR_{i1}^1) - \sigma_2\kappa_{it}\text{Cov}_t(dR_t, dR_{i2}^2)\right]dt \quad (11)
$$

where the three factors are the return of the market and the returns of the two benchmark portfolios, and the asset return covariances with the three factors are equal to $\Sigma_t^{S'}\Sigma_t^{S}1$, $\Sigma_t^{S'}\Sigma_t^{S}\chi_1$ and $\Sigma_t^{S'}\Sigma_t^{S}\chi_2$, respectively.

Proposition 1 shows that asset managers receive a lower expected return for holding assets that have higher covariance with their benchmarks. Intuitively, assets that have higher covariance with their benchmarks are more desirable for the asset managers, as they provide a good hedge against fluctuations in their benchmark level. Therefore, asset managers are happy to receive a lower compensation for the fundamental risk they are taking by holding these assets.
The intuitive formulation of the market price of risk allows us to analyze how cashflow news propagates to the wealth of asset managers. In the next proposition, we characterize their equilibrium risk exposures \( \sigma_{W_i}^t \equiv \sum_t \pi_{it} \).

**Proposition 2 (Asset Managers’ Risk Exposure).** The risk exposure at time \( t \) of asset manager \( i \) is given by the vector

\[
\sigma_{W_i}^t = \nu + \sigma \left( \frac{b_i I_{it} e^{\mu_i(T-t)}}{1 + b_i I_{it} e^{\mu_i(T-t)}} - \kappa_{it} \right) \chi_i - \sigma \kappa_{jt} \chi_j,
\]

for \( i \neq j = 1, 2 \). Consequently, in equilibrium an asset manager is positively exposed to shocks affecting the asset she specializes in, and she is negatively exposed to shocks to the asset the other manager specializes in if and only if the other manager’s benchmark is sufficiently specialized, \( \alpha_j > \bar{\nu}/\sigma \kappa_{jt} \).

For a given set of prices, or Sharpe ratios \( \theta_t \), the optimal risk exposure of asset manager \( i \) to cashflow news in the economy is equal to

\[
\sigma_{W_i}^t = \theta_t + \sigma H_{it} \chi_i,
\]

where

\[
H_{it} = \frac{b_i I_{it} e^{\mu_i(T-t)}}{1 + b_i I_{it} e^{\mu_i(T-t)}}
\]

represents the risk exposure induced by benchmarking. Intuitively, manager \( i \) finds it optimal to tilt the composition of her portfolio towards assets in her benchmark, in order to hedge against fluctuations in the value of her benchmark. Accordingly, \( H_{it} \) captures the additional risk exposure associated with the hedging demand in her portfolio. Therefore, in partial equilibrium, the risk exposure of manager \( i \) is affected only by benchmark \( i \), and not by the other benchmark in the economy.

In general equilibrium, instead, the risk exposure of manager \( i \) becomes dependent on benchmark \( j \) through the endogenous market price of risk \( \theta_t \). Substituting (10) into (13), we obtain (12). The equilibrium risk exposure of manager \( i \) provides valuable insights for the underlying equilibrium mechanism.

First, the higher demand for assets in benchmark \( i \), due to relative performance incentives of manager \( i \), pushes their prices up and consequently their Sharpe ratios down by \( \sigma \kappa_{it} \chi_i \), thus making them less desirable to hold. As a result, the total risk exposure of manager \( i \) to assets in benchmark \( i \) reduces precisely by the amount \( \sigma \kappa_{it} \chi_i \), but remains always positive because of the dominant effect of his hedging.
Second, the higher demand for assets in benchmark \( j \), due to relative performance incentives of manager \( j \), pushes their prices up and consequently their Sharpe ratios down by \( \sigma \kappa_{jt} \chi_j \). As a result, manager \( i \) reduces in equilibrium her risk exposure to the assets in benchmark \( j \) by the amount \( \sigma \kappa_{jt} \chi_j \). Now, suppose benchmark \( j \) receives positive cashflow news. Given her benchmark-induced incentives, asset manager \( j \) increases her demand for assets in benchmark \( j \), thus pushing their prices up. In equilibrium, the additional demand of asset manager \( j \) is “satisfied” by asset manager \( i \), who is induced by market prices to reduce her exposure in benchmark \( j \).

Therefore, the hedging demand of one manager increases the prices of the assets in her benchmark, thus making these assets effectively “overvalued” from the perspective of the other manager. For instance, from the perspective of manager 1, assets outside her benchmark (asset 2 and asset 4) should have the same price-dividend ratio. However, because asset 2 belongs to the benchmark of manager 2, its price-dividend ratio is higher in equilibrium, as (8) reveals. This relative overvaluation is key for the understanding of the economic mechanism responsible for the comovement of asset prices.

Our framework can easily accommodate the presence of retail investors that are not subject to benchmarking concerns. The following remark highlights how to incorporate this feature into the equilibrium presented in Theorem 1.

**Remark 1 (Adding Retail Investors).** A retail investor who is not subject to benchmarking, denoted by subscript \( u \), is a special case of an asset manager with \( b_u = 0 \). If \( \lambda_u = 1 - \sum_{i=1}^{2} \lambda_i \) denotes the initial share of the economy owned by a representative retail investor, the price-dividend ratios and the market price of risk are as in (8) and (10) where now

\[
\kappa_{it} = \lambda_i \frac{b_i I_{it} e^{(\mu_i \alpha_i - \nu i \sigma_i^2)(T-t)}}{1 + b_i I_{i0} e^{\mu_i T}} \left( \frac{\lambda_u}{1 + b_i I_{i0} e^{\mu_i T}} + \sum_{j=1}^{2} \lambda_j \frac{b_j I_{jt} e^{(\mu_j \alpha_j - \nu j \sigma_j^2)(T-t)}}{1 + b_j I_{j0} e^{\mu_j T}} \right)^{-1},
\]

for \( i = 1, 2 \). Our findings remain valid in the presence of retail investors.

### 4 Asset Price Comovements

In this section we study the equilibrium implications for asset price comovements. We demonstrate how heterogeneous benchmarking endogenously generates a mechanism through which fundamental shocks propagate across assets. A notable finding is that, despite independent asset fundamentals,

[^6]: One can easily show that \( H_{it} - \kappa_{it} > 0 \).
heterogeneous benchmarking may give rise to negative asset return correlation. An asset that is included in a benchmark can not only be negatively correlated with assets included in a different benchmark, but also with assets belonging to the same benchmark. We further show that the presence of different benchmarks in the economy triggers additional price pressure amplifying return volatility beyond the volatility characterizing an economy with homogenous benchmarking.

The next proposition characterizes the equilibrium asset return exposures to cashflow news.

**Proposition 3 (Return Exposures).** The equilibrium return exposures to cashflow news at time \( t \) for the assets included in the benchmarks are equal to

\[
\sigma_{1t}^S = \sigma_1^D + \sigma \left( \frac{f_t(\alpha_1) \left( 1 - \sum_{i=1}^{2} \kappa_{it} \right)}{1 + \kappa_{1t} f_t(\alpha_1)} \right) \left( \kappa_{1t} \chi_1 + \frac{\kappa_{1t}\kappa_{2t}}{1 - \sum_{i=1}^{2} \kappa_{it}} (\chi_1 - \chi_2) \right), \tag{15}
\]

\[
\sigma_{2t}^S = \sigma_2^D + \sigma \left( \frac{f_t(\alpha_2) \left( 1 - \sum_{i=1}^{2} \kappa_{it} \right)}{1 + \kappa_{2t} f_t(\alpha_2)} \right) \left( \kappa_{2t} \chi_2 - \frac{\kappa_{1t}\kappa_{2t}}{1 - \sum_{i=1}^{2} \kappa_{it}} (\chi_1 - \chi_2) \right), \tag{16}
\]

\[
\sigma_{3t}^S = \sigma_3^D + \sigma \left( \frac{1 - \sum_{i=1}^{2} \kappa_{it}}{1 + \sum_{i=1}^{2} \kappa_{it} f_t(1 - \alpha_i)} \right) \left( \sum_{i=1}^{2} \kappa_{it} f_t(1 - \alpha_i) \chi_i \right.
\]
\[
+ \left. \frac{\kappa_{1t}\kappa_{2t}}{1 - \sum_{i=1}^{2} \kappa_{it}} \left[ f_t(1 - \alpha_1) - f_t(1 - \alpha_2) \right] (\chi_1 - \chi_2) \right). \tag{17}
\]

The return exposure of the asset not included in the benchmarks is equal to \( \sigma_{4t}^S = \sigma_4^D \). Consequently, heterogeneous benchmarking, \( \chi_2 \neq \chi_1 \), makes positive cashflow news to specialized asset \( i \):

- reduce the return of specialized asset \( j \), for \( i \neq j = 1, 2 \);

- reduce the return of common asset 3 if benchmark \( i \) is more specialized than benchmark \( j \), \( \alpha_i > \alpha_j \), and increase it otherwise, for \( i \neq j = 1, 2 \).

Moreover, heterogeneous benchmarking makes positive cashflow news to common asset 3 reduce the return of specialized asset \( j \) and increase the return of specialized asset \( i \), if benchmark \( i \) is less specialized than benchmark \( j \), \( \alpha_i < \alpha_j \), for \( i \neq j = 1, 2 \).

In a standard economy in which asset managers are not subject to benchmarking, the return of a risky asset is only exposed to its own cashflow news. In an economy in which asset managers are all subject to the same benchmark, instead, the return of a risky asset that is included in the benchmark becomes positively exposed to cashflow news concerning all the risky assets included in the benchmark.
This generates a so-called “asset-class” effect (e.g., Barberis, Shleifer, and Wurgler, 2005; Boyer, 2011). Proposition 3 reveals that when asset managers are each subject to a different benchmark, the return of a specialized asset, which is only included in one of the two benchmarks, becomes exposed to cashflow news concerning the specialized asset of the other benchmark. Formally, besides depending on $\chi_i$, $\sigma^S_{it}$ is also a function of $\chi_j$ when the two benchmarks are heterogeneous, $\chi_2 \neq \chi_1$. Therefore, heterogeneous benchmarking leads to contagion of cashflow news across assets belonging to different benchmarks.

Our analytical characterization of the asset return exposures allows for an explicit decomposition in three parts: (i) a fundamental component; (ii) a component due to homogeneous benchmarking; and (iii) a component capturing the effects of heterogeneous benchmarking. We illustrate this decomposition by means of the following three-matrix representation, which is parameterized below, without loss of generality, for the case in which benchmark 2 is more specialized than benchmark 1, $\alpha_2 > \alpha_1$:

$$
\begin{bmatrix}
\sigma^S_{1t} \\
\sigma^S_{2t} \\
\sigma^S_{3t} \\
\sigma^S_{4t}
\end{bmatrix} =
\begin{bmatrix}
+ & 0 & 0 & 0 \\
- & + & 0 & 0 \\
0 & 0 & + & 0 \\
0 & 0 & 0 & +
\end{bmatrix} +
\begin{bmatrix}
+ & 0 & + & 0 \\
0 & + & + & 0 \\
+ & + & + & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} +
\begin{bmatrix}
+ & + & + & 0 \\
- & + & + & 0 \\
0 & - & + & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

(18)

All three matrices are populated with plusses, minuses, or zeros, representing the signs of the effect of cashflow news about an asset (columns) to all other assets (rows): the entry in position ($k, \ell$) indicates the effect of cashflow news for asset $\ell$ on the return of asset $k$. The first matrix then isolates the fundamental component of the return exposures, the second matrix isolates the component due to homogeneous benchmarking, and the last matrix singles out the component capturing the effects of heterogeneous benchmarking. The sum of three matrices yields the total asset return exposures.

To present the intuition behind the different effects in (18), let us consider positive cashflow news for asset 1 by focusing on the first column of each of the three matrices. To reflect the positive change in asset 1’s fundamentals, the price of asset 1 adjusts upward. This explains the “+” sign in position (1,1) of the first matrix. The second matrix, instead, shows that the presence of benchmark 1 in the economy amplifies the fundamental price increase in asset 1 and triggers a price increase in asset 3. The intuition is as follows. Since asset 1 belongs to benchmark 1, positive news to asset 1’s fundamentals translates into an increase in value of benchmark 1. This in turn makes manager 1 wealthier and increases her desire to be exposed to risky assets. In an attempt not to fall behind her benchmark, she increases her exposure to both assets in her benchmark (asset 1 and asset 3), proportionally to their relative weights ($\alpha_1$ and $1-\alpha_1$). In equilibrium, the associated positive price pressure in both assets included in benchmark 1 is captured by the “+” signs in position (1,1) and (3,1) of the second matrix. This confirms
the findings in Basak and Pavlova (2014), where only one benchmarked asset manager is present in the economy.

How does the presence of manager 2 in the economy affect asset prices in equilibrium? Remember that, because of her hedging demand, manager 2 makes the assets in her benchmark (asset 2 and asset 3) “overvalued” from the perspective of manager 1. For this reason alone, manager 1 has a reduced exposure to asset 2 and asset 3 in equilibrium. Now, going back to the positive cashflow news for asset 1, the increase in wealth of manager 1 induces her to exploit the overvaluation of asset 2 and asset 3. This allows her to further increase her exposure to the assets in her own benchmark. In particular, manager 1 sells asset 2, thus driving its price down, and buys asset 1, thus driving its price up. This explains the “−” sign in position (2,1) and the “+” sign in position (1,1) of the third matrix.

Regarding the common asset, instead, manager 1 faces the following trade-off: on one hand, she wants to sell asset 3 to exploit the overvaluation induced by manager 2; on the other hand, she wants to buy asset 3 to be closer to her benchmark. Equation (17) reveals that whether manager 1 will ultimately buy or sell asset 3 in equilibrium only depends on the relative degree of specialization of the two benchmarks. Intuitively, from the perspective of manager $i$, $(1 - \alpha_i)$ captures the importance of holding the common asset to be close to her benchmark, whereas $(1 - \alpha_j)$ captures the importance of the common asset’s overvaluation induced by manager $j$. Therefore, when the benchmark of manager 1 has a higher weight on the common asset than benchmark 2, $1 - \alpha_1 > 1 - \alpha_2$, benchmarking concerns dominate and she increases her exposure to asset 3. This confirms the “+” sign in position (3,1) of the third matrix. The same intuition validates the “−” sign in position (3,2) of the third matrix. Following positive cashflow news for asset 2, the importance for manager 2 to capture the common asset’s overvaluation dominates her desire to stay close to her benchmark, thus inducing her to sell asset 3.\(^7\) This highlights an asymmetry in how shocks to specialized assets propagate differently to asset that are included in multiple benchmarks.

Note that in order to understand how heterogenous benchmarking affects equilibrium prices following cashflow news, it suffices to analyze the price pressures induced by the manager who becomes wealthier in relative terms. Accordingly, a positive shock to asset 3’s fundamentals triggers a wealth redistribution in favor of the manager whose benchmark has a higher weight in it, and hence is less specialized. In equilibrium, this manager demands more of her specialized asset, less of the other benchmark’s specialized asset, and more of the common asset. That she demands more of the common asset is the case because her benchmarking concerns always outweigh the incentives to exploit the overvaluation of

\(^7\)Note that when the two benchmarks are equally specialized, $\alpha_2 = \alpha_1$, the incentives to capture overvaluation are perfectly offset by benchmarking concerns. In this special case, heterogeneous benchmarking does not affect the equilibrium price of the common asset.
that asset. Given the parametrization $\alpha_2 > \alpha_1$ in (18), the prices of specialized asset 1 and common asset 3 increase, whereas the price of specialized asset 2 decreases. Accordingly, positions (1,3) and (3,3) of the third matrix feature a “+” sign, whereas position (2,3) features a “−” sign.

In sum, heterogeneous benchmarking is responsible for asset contagion and negative spillovers of fundamental shocks across risky assets that make up benchmarks. This novel mechanism provides a rich set of implications on asset correlation and volatility, which we present and discuss in what follows.

### 4.1 Asset Correlation

In this section we study the asset return correlations that are implied by the equilibrium return exposures. We define the (instantaneous) correlation between the returns of asset $k$ and asset $\ell$ at time $t$ as

$$
\rho_t^S(dR_{kt}, dR_{\ell t}) \equiv \frac{\sigma_{kt}^S \sigma_{\ell t}^S}{\sqrt{(\sigma_{kt}^S \sigma_{kt}^S) (\sigma_{\ell t}^S \sigma_{\ell t}^S)}}
$$

The next proposition introduces our key findings on asset correlation.

**Proposition 4 (Return Correlation).** Suppose benchmark 2 is more specialized than benchmark 1, $\alpha_2 \geq \alpha_1$. It follows that the return correlation between specialized asset 1 and the common asset is always positive, $\rho_t^S(dR_{1t}, dR_{3t}) \geq 0$. Moreover, if $\sigma^2 \leq \log 2/T$, the following holds:

- **the return correlation between the two specialized assets is negative**, $\rho_t^S(dR_{1t}, dR_{2t}) \leq 0$;
- **the return correlation between specialized asset 2 and the common asset is negative**, $\rho_t^S(dR_{2t}, dR_{3t}) \leq 0$, when the degree of specialization of benchmark 2 is sufficiently high, $\alpha_2 \geq \overline{\alpha_2}$, and is positive, $\rho_t^S(dR_{2t}, dR_{3t}) \geq 0$, when it is sufficiently low, $\alpha_2 < \underline{\alpha_2}$, where $0 \leq \underline{\alpha_2} \leq \overline{\alpha_2} \leq 1$.

Notwithstanding its simplicity, our setting yields a broad set of implications for asset comovements; not just for assets within the same benchmark, but also for assets across benchmarks. In particular, our analysis of asset correlation across benchmarks would not be possible in a setting with homogeneous benchmarking, where only within benchmark comovements can be studied.

The key result in Proposition 4 is that asset return correlation can become negative in equilibrium. Notably, negative return correlation occurs not only across benchmarks, but also within benchmarks.
In this figure we plot the equilibrium return correlation (in percentage) between asset $k$ and $\ell$, $\rho_{kl}^S(dR_{kt}, dR_{\ell t})$ as a function of the degree of specialization of benchmark 2, $\alpha_2$. The parameter values are $\lambda_i = 0.5$, $b_i = 1$, $\alpha_1 = 0.3$, $\mu = 5\%$, $\sigma = 18\%$, $\nu = \sigma/5$, $t = 1$, $T = 5$, $D_{k0} = 1$, $D_{kt} = 2$, for $i = 1, 2$ and $k = 1, 2, 3, 4$. The plots are typical.

This is a consequence of the negative return exposures, which, as we discussed above, only arise in the presence of heterogeneous benchmarking. To guide our examination of asset correlation within and across benchmarks, Figure 1 presents the pairwise correlations in our economy as a function of degree of specialization of benchmark 2.

The left panel of Figure 1 depicts the comovements of asset returns across benchmarks (specialized asset 1 and specialized asset 2). It shows that asset return correlation across benchmarks is negative and therefore below the correlation of their fundamentals, which in our setting is intentionally set to zero. The plot further reveals that the extent of the deviation from cashflow correlation is increasing in the degree of specialization of the more specialized benchmark in the economy, benchmark 2. To understand the intuition underlying this result, remember from the third component of the exposure decomposition that any cashflow news to asset 1, 2 or 3 induces the managers to rebalance their portfolios in opposite directions. When manager 1 increases her exposure to benchmark 1, manager 2 decreases her exposure to benchmark 2, and vice versa. This makes the specialized assets of both benchmarks negatively
correlated. For a given $\alpha_1$, the mechanism is stronger when the degree of specialization of benchmark 2, $\alpha_2$, is higher, since in that case, altering the exposure to benchmark 2 entails stronger price pressure on asset 2.

Negative asset return comovements across benchmarks are in line with weakened comovements across asset styles that Barberis and Shleifer (2003) obtain. In their behavioral model, investors move funds across investment styles, chasing performance based on exogenous rules. We, instead, derive our results in a fully rational equilibrium model in which managers care about their performance relative to designated benchmarks, capturing their investment styles.

Proposition 4 further shows that assets within the same benchmark can also be negatively correlated. Specifically, this can occur within the more specialized benchmark in the economy, if it is sufficiently tilted towards its specialized asset. The intuition goes as follows. A wealth redistribution in the economy (following cashflow news) implies that the price pressure on the common asset always follows the price pressure induced by the manager with the less specialized benchmark. When $\alpha_2 > \alpha_1$, this further implies that heterogeneous benchmarking increases the comovement of asset 1 and 3, and reduces the comovement of asset 2 and 3. Since these effects are stronger when the benchmarks have polarized weights (i.e., $\alpha_2 \gg \alpha_1$), this explain why negative return correlation between asset 2 and 3 arises when $\alpha_2$ is sufficiently high ($\alpha_2 > \overline{\alpha}_2$). The increased comovement of asset 1 and 3, instead, amplifies the standard asset-class effect of an economy with homogenous benchmarking. The middle and the right panels of Figure 1 depict these findings.\(^8\)

### 4.2 Asset Volatility

In this section we discuss the equilibrium implications for asset return volatility, emphasizing the effects of heterogeneous benchmarking. We define the (instantaneous) volatility of asset $k$ at time $t$ as

$$\var_t(dR_{kt}) \equiv \sigma_{kt}$$

The next proposition introduces our key findings on asset volatility.

---

\(^8\)The middle panel in Figure 1 also shows that the correlation between specialized asset 2 and the common asset increases in $\alpha_2$ if $\alpha_2$ is sufficiently low. This is because when $\alpha_2$ is low the effects of heterogeneous benchmarking are small, making the standard amplification effects associated with homogenous benchmarking (second matrix of our proposed decomposition in (18)) dominate.
In this figure we plot the equilibrium non-fundamental return volatility (in percentage) of asset $k$, $\sqrt{\text{Var}_t(dR_{kt})} - \sigma^2$ as a function of the degree of specialization of benchmark 2, $\alpha_2$. The parameter values are as in Figure 1. The plots are typical.

**Proposition 5 (Return Volatility).** Suppose $I_{1t} = I_{2t}$ and $b_1 = b_2$. It follows that specialized asset 2 is more volatile than specialized asset 1, $\text{Var}_t(dR_{2t}) \geq \text{Var}_t(dR_{1t})$, iff benchmark 2 is more specialized than benchmark 1, $\alpha_2 \geq \alpha_1$. Moreover, if $\sigma^2 \leq \log 2/T$, the following holds:

- specialized asset $i$ is more volatile than the common asset, $\text{Var}_t(dR_{it}) \geq \text{Var}_t(dR_{3t})$, when the degree of specialization of benchmark $i$ is sufficiently high, $\alpha_i \geq \bar{\alpha}_i^*$, and it is less volatile, $\text{Var}_t(dR_{it}) \leq \text{Var}_t(dR_{3t})$ when the degree of specialization is sufficiently low, $\alpha_i \leq \underline{\alpha}_i^*$;

- specialized asset $i$ can be more volatile than the common asset despite the sum of the benchmarks’ weights on the common asset being larger than the weight on the specialized asset $i$, $\sum_{j=1}^{2}(1-\alpha_j) \geq \alpha_i$. 

---

**Figure 2: Asset Volatility**

The figure illustrates the relationship between asset volatility and the degree of specialization of benchmark 2 ($\alpha_2$). The plots show the equilibrium non-fundamental return volatility as a function of $\alpha_2$. The parameter values used are as in Figure 1, and the plots are typical for the given parameter configuration.
Existing models with homogenous benchmarking (Cuoco and Kaniel, 2011; Basak and Pavlova, 2013), deliver the result that assets that are included in a benchmark have higher return volatility. A natural implication is that assets with higher weight in the benchmark have even higher volatility. The second matrix of our asset exposure decomposition in (18) confirms these results. Proposition 5, however, reveals that heterogeneous benchmarking may alter the implicit link between benchmark weights and price pressures, and consequently between benchmark weights and volatilities.

Consider for example the case in which the benchmarked assets 1, 2 and 3 are overall equally weighted across benchmarks: say \( \alpha_1 = \frac{2}{3}, \alpha_2 = \frac{2}{3} \), and \( (1 - \alpha_1) + (1 - \alpha_2) = \frac{2}{3} \). It may seem natural for the three assets to have exactly the same volatilities in this case. This would indeed be true if the three assets were included with these weights in a homogenous benchmark. However, this does not hold in our economy with heterogenous benchmarking, where instead the return volatility of the common asset is lower:

\[
\text{Var}_t(dR_{3t}) < \text{Var}_t(dR_{1t}) = \text{Var}_t(dR_{2t}).
\]

In this example, the opposing price pressures of the two managers on asset 3 perfectly cancel out, thus leaving only asset 1 and 2 affected by the amplification effect of heterogeneous benchmarking.

More generally, our findings highlight how in equilibrium the common asset may be less volatile than a specialized asset, even when its total weight across benchmarks, \( \sum_{j=1}^{2} (1 - \alpha_j) \), exceeds the weight of the specialized asset, \( \alpha_i \). Figure 2 visualizes this results by highlighting the values of \( \alpha_2 \) for which the return volatility of the common asset is lower than the return volatility of specialized asset 2, despite \( (1 - \alpha_1) + (1 - \alpha_2) \) being larger than \( \alpha_2 \). This specific region of \( \alpha_2 \)-values corresponds to the “triangular” region delimited above by the red line, below by the black line, and on the right by the vertical dotted line corresponding to \( \alpha_2 = 1 - \alpha_1/2 \).

5 Concluding Remarks

In this paper we study and explicitly characterize the equilibrium implications of heterogeneous benchmarking. We consider an economy in which asset managers are each subject to a different benchmark, and solve for equilibrium asset prices and their properties. Our framework is tractable and allows us to characterize the equilibrium quantities in closed-form.

Our analysis demonstrates that heterogeneous benchmarking can lead to contagion of cashflow news across assets belonging to different benchmarks. Accordingly, we characterize a rich structure of asset price comovements that delivers new testable implications. Despite independent asset fundamentals,
heterogeneous benchmarking may give rise to negative asset return correlation. We show that an asset that is included in a benchmark can not only be negatively correlated with assets included in a different benchmark, but also with assets belonging to the same benchmark. Negative asset return comovements across benchmarks are in line with documented weakened comovements across investment styles. Moreover, the presence of different benchmarks in the economy triggers additional price pressure amplifying return volatility beyond the levels characterizing an economy in which all benchmarks are identical.
Appendix A: Proofs

We define below auxiliary quantities that are used throughout the proofs.

Definition A.1 (Auxiliary quantities).

\[ w_{it} \equiv A_i b_i I_{it} e^{(\mu_i - \nu_i')(T-t)}, \]  
\[ c_t \equiv \sum_{i=1}^{2} (A_i + w_{it}) , \]  
\[ \dot{C}_{it} = c_t (c_t + w_{it} f_t(\alpha_i)) , \]  
\[ C_{3t} = c_t \left( c_t + \sum_{i=1}^{2} w_{it} f_t(1-\alpha_i) \right) , \]  
\[ \kappa_{it} = \frac{w_{it}}{c_t} \]  

where \( i = 1, 2 \).

Proof of Theorem 1. As markets are dynamically complete in our economy, we can solve the dynamic optimization problem of each asset manager using the martingale method (Karatzas, Lehoczky and Shreve, 1987; Cox and Huang, 1989). This allows us to equivalently define the set of feasible policies \( \pi_{it} \) in terms of the horizon wealth \( W_{iT} \) subject to the following static budget constraint \( \mathbb{E}[M_T W_{iT}] / \mathbb{E}[M_T] = \lambda_i S_{m0} \), where, \( S_{m0} \) is initial value of the aggregate asset market, \( S_{m0} = \sum_{k=1}^{N+1} S_{k0} \), and \( M_T \) is the pricing kernel. Given prices, the first order condition of asset manager \( i \) is given by

\[ \mathbb{E}[M_T W_{iT}] = \frac{1 + b_i I_{iT}}{\mathbb{E}[1 + b_i I_{iT}]} \lambda_i S_{m0} = A_i (1 + b_i I_{iT}) S_{m0} , \]  

(A.6)

Let us conjecture the sharing rule \( P_{iT} \),

\[ P_{iT} = \frac{A_i (1 + b_i I_{iT})}{\sum_{i=1}^{2} A_i (1 + b_i I_{iT})} , \]  

(A.7)

where the constant \( A_i \) is characterized by

\[ A_i = \frac{\lambda_i}{\mathbb{E}[1 + b_i I_{iT}]} , \]  

(A.8)

Each institution consumes a fraction \( P_{iT} \) of the aggregate dividend. There are two conditions \( P_{iT} \) has to meet. First, shares are non-negative, \( P_{iT} \geq 0 \), and sum to one, \( \sum_{i=1}^{2} P_{iT} = 1 \). These two conditions
are straightforward to verify from the formulation. Second, by consuming $P_i T D_T$, at maturity, market participants are maximizing their terminal wealth.

$$P_i T D_T = \frac{S_{m0} A_i (1 + b_i I_{iT})}{S_{m0} \sum_{j=1}^{2} A_j (1 + b_j I_{jT})} D_T = \frac{M_T \mathbb{E}[M_T] W_{iT}}{S_{m0} \sum_{j=1}^{2} A_j (1 + b_j I_{jT})} D_T = \frac{M_T \mathbb{E}[M_T] W_{iT} D_T}{\sum_{j=1}^{2} M_T \mathbb{E}[M_T] W_{jT}} D_T = \frac{W_{iT}}{\sum_{j=1}^{2} W_{jT}} D_T = W_{iT}.$$  

The first equality is by plugging the conjectured risk sharing, and dividing and multiplying by the total asset market $S_{m0}$, the second is by using the first order condition for institution $i$, the third is by using the first order condition for institution $j$, and the last by observing that consumption good markets clear. The pricing kernel $M_T$ is obtained by summing the first order conditions, (A.6), of the markets participants, and clearing the consumption good market

$$M_T = D_T^{-1} \sum_{i=1}^{2} A_i (1 + b_i I_{iT}).$$

By the no arbitrage condition in a complete market economy like this one, we know that

$$S_{kt} = \frac{\mathbb{E}_t [M_T D_{kt}]}{\mathbb{E}_t [M_T]}.$$  

Because all the random variables in this setting are geometric brownian motions, we can solve the expectations such that

$$\mathbb{E}_t [M_T] = \mathbb{E}_t \left[ D_T^{-1} \sum_{i=1}^{2} A_i (1 + b_i I_{iT}) \right]$$

$$= D_T^{-1} \sum_{i=1}^{2} A_i e^{(-\eta + \|\nu\|^2)(T-t)} + D_T^{-1} \sum_{i=1}^{2} b_i A_i I_{iT} e^{(-\eta + \|\nu\|^2 + \mu I_i - \nu I_i') (T-t)},$$  

$$\mathbb{E}_t [M_T D_{kt}] = D_T^{-1} D_{kt} \sum_{i=1}^{2} A_i e^{(-\eta + \|\nu\|^2 + \mu - \nu' \sigma_k) (T-t)}$$

$$+ D_T^{-1} D_{kt} \sum_{i=1}^{2} b_i A_i I_{iT} e^{(-\eta + \|\nu\|^2 + \mu I_i - \nu' \sigma_k' + \mu D_k - \nu' \sigma_k' + \sigma D_k' \sigma_i') (T-t)},$$  

$$\mathbb{E}_t [M_T P_{iT} D_T] = \mathbb{E}_t [A_i (1 + b_i I_{iT})] = A_i + A_i b_i I_{iT} e^{\mu I_i (T-t)}.$$  

22
Dividing (A.10) by (A.9) and canceling similar terms we get

\[ S_{kt} = D_{kt} \sum_{i=1}^{2} A_i e^{\left(\mu - \nu \sigma^{D_k} - \nu \sigma^{D_k} (T-t)\right)} \]

\[ \frac{\sum_{i=1}^{2} A_i + \sum_{i=1}^{2} b_i A_i I_{it} e^{\left(\mu - \nu \sigma^{D_k} + \sigma^{D_k} (T-t)\right)}}{\sum_{i=1}^{2} A_i + \sum_{i=1}^{2} b_i A_i I_{it} e^{\left(\mu - \nu \sigma^{D_k} + \sigma^{D_k} (T-t)\right)}}. \]

Using Definition (A.1), and (A.3) we get

\[ S_{3t} = D_{3t} e^{\left(\mu - \nu \sigma^{D_k} - \nu \sigma^{D_k} (T-t)\right)} \frac{\sum_{i=1}^{2} A_i + \sum_{i=1}^{2} w_{it} e^{\left(\sigma^{D_k} - \sigma^{D_k} (T-t)\right)}}{\sum_{i=1}^{2} A_i + \sum_{i=1}^{2} w_{it}} \]

\[ = D_{3t} e^{\left(\mu - \nu \sigma^{D_k} - \nu \sigma^{D_k} (T-t)\right)} c_t + \sum_{i=1}^{2} w_{it} e^{\left(\sigma^{D_k} - \sigma^{D_k} (T-t)\right) - 1} c_t \]

\[ = D_{3t} e^{\left(\mu - \nu \sigma^{D_k} - \nu \sigma^{D_k} (T-t)\right)} \left(1 + \sum_{i=1}^{2} w_{it} f_t (1 - \alpha_1)\right). \]

Obtaining \( S_{0t}, S_{1t} \) and \( S_{2t} \) is done the same way. The market prices of risk are obtained by applying Itô’s lemma to \( \frac{\mathbb{E}_t [M_T]}{\mathbb{E} [M_T]} \), characterized by

\[ d \left( \frac{\mathbb{E}_t [M_T]}{\mathbb{E} [M_T]} \right) = - \left( \frac{\mathbb{E}_t [M_T]}{\mathbb{E} [M_T]} \right) \theta_t = \left( \frac{\mathbb{E}_t [M_T]}{\mathbb{E} [M_T]} \right) \left( -\nu + \sum_{i=1}^{2} \sigma^{I_i} \frac{w_{it}}{c_t} \right) \quad (A.12) \]

Plugging the definition of \( \sigma^{I_i} \) we get the desired result.

The last equality is obtained by applying Itô’s Lemma to \( I_{it} = D_{it}^{\alpha_i} D_{3t}^{1-\alpha_i} \), where the drift and volatility coefficients are characterized by

\[ \mu^{I_i} = \mu - \sigma^2 \chi_1 (1 - \alpha_i), \quad (A.13) \]

\[ \sigma^{I_1} = \sigma \chi_1, \quad \sigma^{I_2} = \sigma \chi_2. \quad (A.14) \]

\[ \square \]

**Proof of Proposition 1.** To be completed.

**Proof of Proposition 2.** Wealth volatility is obtained by applying Itô’s lemma to \( W_{it} E_t [M_T] = E_t [M_T P_{it} D_T] \) and comparing the volatility terms. Comparing volatility terms we obtain

\[ \Sigma_s^{S} \bar{\pi}_{it} + \frac{d E_t [M_T]}{E_t [M_T]} = \frac{d E_t [M_T P_{it} D_T]}{E_t [M_T P_{it} D_T]}. \]
Plugging \( \theta_t \), (A.12), and (A.11) we obtain

\[
\Sigma_t^{S_{it}} = \theta_t + \frac{d \left( A_i + A_i b_i I_{it} e^{\mu_i (T-t)} \right)}{A_i + A_i b_i I_{it} e^{\mu_i (T-t)}} = \theta_t + \frac{b_i I_{it} e^{\mu_i (T-t)}}{1 + b_i I_{it} e^{\mu_i (T-t)}} \sigma^I_t.
\]

Plugging the definition of \( \sigma^I_t \), and \( \theta_t \) we get the desired result. \( \square \)

**Proof of Proposition 3.** The asset return exposures are obtained by applying Itô’s lemma to \( S_{kt} \mathbb{E}_t [M_T] = \mathbb{E}_t [M_T D_{kT}] \) and comparing volatility terms. Doing so we obtain the equation characterized by

\[
\frac{dS_{kt}}{S_{kt}} + \frac{d \mathbb{E}_t [M_T]}{\mathbb{E}_t [M_T]} + \left( \frac{dS_{kt}}{S_{kt}} \right) \left( \frac{d \mathbb{E}_t [M_T]}{\mathbb{E}_t [M_T]} \right) = \frac{d \mathbb{E}_t [M_T D_{kT}]}{\mathbb{E}_t [M_T D_{kT}]}.
\]

We know that the first volatility of \( dS_{kt}/S_{kt} \) is \( \sigma^S_{kt} \), and from (A.12) we obtain the volatility term of \( d(\mathbb{E}_t [M_T])/\mathbb{E}_t [M_T] \). Plugging these relations we obtain

\[
\sigma^S_{kt} = \theta_t + \frac{d \mathbb{E}_t [M_T D_{kT}]}{\mathbb{E}_t [M_T D_{kT}]}
= \theta_t + \sigma^{D_k} - \nu + \sum_{i=1}^{2} \sigma^I_i \frac{b_i A_i I_{it} e^{(\mu^I_i - \nu \sigma^I_i + \sigma^{D_{k'}} \sigma^I_i) (T-t)}}{\sum_{j=1}^{2} A_j + \sum_{j=1}^{2} b_j A_j I_{jt} e^{(\mu^I_j - \nu \sigma^I_j + \sigma^{D_{k'}} \sigma^I_j) (T-t)}}.
\]

Plugging \( \theta_t \), (10), and \( C_{kt} \), (A.3), we obtain

\[
\sigma^S_{kt} = \sigma^{D_k} - \nu + \nu - \sum_{i=1}^{2} \sigma^I_i \frac{w_{it}}{c_t} + \sum_{i=1}^{2} \sigma^I_i \frac{b_i A_i I_{it} e^{(\mu^I_i - \nu \sigma^I_i + \sigma^{D_{k'}} \sigma^I_i) (T-t)}}{\sum_{j=1}^{2} A_j + \sum_{j=1}^{2} b_j A_j I_{jt} e^{(\mu^I_j - \nu \sigma^I_j + \sigma^{D_{k'}} \sigma^I_j) (T-t)}}
= \sigma^{D_k} - \sum_{i=1}^{2} \sigma^I_i \frac{w_{it}}{c_t} + \sum_{i=1}^{2} \sigma^I_i \frac{b_i A_i I_{it} e^{(\mu^I_i - \nu \sigma^I_i + \sigma^{D_{k'}} \sigma^I_i) (T-t)}}{\sum_{j=1}^{2} A_j + \sum_{j=1}^{2} b_j A_j I_{jt} e^{(\mu^I_j - \nu \sigma^I_j + \sigma^{D_{k'}} \sigma^I_j) (T-t)}}
= \sigma^{D_k} - \sum_{i=1}^{2} \sigma^I_i \frac{w_{it}}{c_t} + \sum_{i=1}^{2} \sigma^I_i \frac{w_{it} \left( e^{(\sigma^{D_{k'}} \sigma^I_i) (T-t)} - 1 \right) + w_{it}}{\sum_{j=1}^{2} (A_j + w_{jt} + w_{jt} \left( e^{(\sigma^{D_{k'}} \sigma^I_j) (T-t)} - 1 \right))}.
\]
Specializing the result to asset of type 3,

\[ \sigma_{3t}^S = \sigma_{3t}^D + \sum_{i=1}^{2} \sigma_{i}^{l_i} \left( \frac{w_{it} f_t (1 - \alpha_i) + w_{it}}{c_t + \sum_{j=1}^{2} w_{jt} f_t (1 - \alpha_j)} - \frac{w_{it}}{c_t} \right) \]

\[ = \sigma_{3t}^D + \sum_{i=1}^{2} \frac{c_t w_{it} f_t (1 - \alpha_i) + c_t w_{it} - w_{it} c_t - w_{it} \sum_{j=1}^{2} w_{jt} f_t (1 - \alpha_j)}{C_{3t}} \]

\[ = \sigma_{3t}^D + \sum_{i=1}^{2} \frac{c_t w_{it} f_t (1 - \alpha_i) - w_{it} \sum_{j=1}^{2} w_{jt} f_t (1 - \alpha_j)}{C_{3t}} \]

\[ = \sigma_{3t}^D + \frac{1}{C_{3t}} \left[ \sum_{i=1}^{2} \sigma_{i}^{l_i} (c_t w_{it} f_t (1 - \alpha_i)) - \sum_{i=1}^{2} \sigma_{i}^{l_i} \left( w_{it} \sum_{j=1}^{2} w_{jt} f_t (1 - \alpha_j) \right) \right]. \]

Plugging \( c, (A.3) \), and rearranging we get

\[ \sigma_{3t}^S = \sigma_{3t}^D + \sum_{i=1}^{2} \sigma_{i}^{l_i} (w_{it} f_t (1 - \alpha_i)) + \frac{1}{C_{3t}} \sum_{i=1}^{2} \sigma_{i}^{l_i} (w_{it} f_t (1 - \alpha_i)) \left( \sum_{j=1}^{2} w_{jt} \right) \]

\[ - \sum_{i=1}^{2} \sigma_{i}^{l_i} \left( w_{it} \sum_{j=1}^{2} w_{jt} f_t (1 - \alpha_j) \right) \]

\[ = \sigma_{3t}^D + \sum_{i=1}^{2} \frac{A_i}{C_{3t}} \sum_{i=1}^{2} \sigma_{i}^{l_i} (w_{it} f_t (1 - \alpha_i)) + \frac{1}{C_{3t}} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{i}^{l_i} w_{it} w_{jt} (f_t (1 - \alpha_i) - f_t (1 - \alpha_j)). \]

The volatilities of asset 1 and asset 2 are obtained the same way. \( \square \)

**Lemma A.1.** The correlation between two asset types is characterized by

\[
\frac{\sigma_{kt}^S \sigma_{lt}^S}{\sigma^2} = 1_k' 1_l \\
+ (\Omega_{kt}^1 \chi_k^1 + \Omega_{kt}^2 \chi_k^2 + \Omega_{kt}^1 \chi_l^1 + \Omega_{kt}^2 \chi_l^2) \\
+ (\Omega_{kt}^1 \Omega_{kt}^1 [(\alpha_1)^2 + (1 - \alpha_1)^2] + \Omega_{kt}^2 \Omega_{kt}^2 [(\alpha_2)^2 + (1 - \alpha_2)^2]) \\
+ (\Omega_{kt}^1 \Omega_{kt}^2 + \Omega_{kt}^1 \Omega_{kt}^2) (1 - \alpha_1) (1 - \alpha_2),
\]

(A.15)

where

\[ \Omega_{kt}^1 = D_{kt}^1 + E_{kt} = \frac{1}{C_{kt}} (A_1 + A_2) (w_{1t} f_1 (\alpha_1) \chi_k^1) + \frac{1}{C_{kt}} w_{1t} w_{2t} (f_1 (\alpha_1) \chi_k^1 - f_2 (\alpha_2) \chi_k^2), \]

(A.16)

\[ \Omega_{kt}^2 = D_{kt}^2 - E_{kt} = \frac{1}{C_{kt}} (A_1 + A_2) (w_{2t} f_2 (\alpha_2) \chi_k^2) - \frac{1}{C_{kt}} w_{1t} w_{2t} (f_1 (\alpha_1) \chi_k^1 - f_2 (\alpha_2) \chi_k^2), \]

(A.17)
for \( k = 1, 2, \) and
\[
\Omega_{3t}^1 = D_{kt}^1 + E_{kt} = \frac{1}{C_{3t}} (A_1 + A_2) (w_{1t} f (1 - \alpha_1) \chi_3^1) + \frac{1}{C_{kt}} w_{1t} w_{2t} (f (1 - \alpha_1) \chi_3^1 - f (1 - \alpha_2) \chi_3^2),
\]
\[
\Omega_{3t}^2 = D_{kt}^2 - E_{kt} = \frac{1}{C_{3t}} (A_1 + A_2) (w_{2t} f (1 - \alpha_2) \chi_3^2) - \frac{1}{C_{kt}} w_{1t} w_{2t} (f (1 - \alpha_1) \chi_3^1 - f (1 - \alpha_2) \chi_3^2),
\]
(A.18)

Specializing the result to different types of assets we get
\[
\frac{\sigma_{3}^{S_1} \sigma_{3}^{S_2}}{\sigma^2} = (2\Omega_{3t}^1 (1 - \alpha_1) + 2\Omega_{3t}^2 (1 - \alpha_2)) + J_{33}^{within type} + J_{33}^{across types},
\]
(A.20)
\[
\frac{\sigma_{1}^{S_1} \sigma_{3}^{S_2}}{\sigma^2} = (\Omega_{1t}^1 \alpha_1 + \Omega_{1t}^2 (1 - \alpha_1) + \Omega_{2t}^2 (1 - \alpha_2)) + J_{13}^{within type} + J_{13}^{across types},
\]
(A.21)
\[
\frac{\sigma_{2}^{S_1} \sigma_{3}^{S_2}}{\sigma^2} = (\Omega_{2t}^2 \alpha_2 + \Omega_{2t}^1 (1 - \alpha_1) + \Omega_{2t}^2 (1 - \alpha_2)) + J_{23}^{within type} + J_{23}^{across types},
\]
(A.22)
\[
\frac{\sigma_{1}^{S_1} \sigma_{2}^{S_2}}{\sigma^2} = 2\Omega_{1t}^1 \alpha_1 + J_{11}^{within type} + J_{11}^{across types},
\]
(A.23)
\[
\frac{\sigma_{2}^{S_1} \sigma_{2}^{S_2}}{\sigma^2} = (\Omega_{2t}^2 \alpha_2 + \Omega_{2t}^1 (1 - \alpha_1) + \Omega_{2t}^2 (1 - \alpha_2)) + J_{12}^{within type} + J_{12}^{across types},
\]
(A.24)
\[
\frac{\sigma_{1}^{S_1} \sigma_{1}^{S_2}}{\sigma^2} = 2\Omega_{2t}^2 \alpha_2 + J_{22}^{within type} + J_{22}^{across types},
\]
(A.25)

where
\[
J_{kt}^{within type} = (\Omega_{kt}^1 \Omega_{kt}^1 [(1 - \alpha_1)^2 + (\alpha_1)^2] + \Omega_{kt}^2 \Omega_{kt}^2 [(1 - \alpha_2)^2 + (\alpha_2)^2]),
\]
\[
J_{kt}^{across types} = (\Omega_{kt}^1 \Omega_{kt}^1 + \Omega_{kt}^2 \Omega_{kt}^2) (1 - \alpha_1) (1 - \alpha_2).
\]

The first element in (A.15) is the fundamental volatility element. The second element is the effect of benchmarking. The third and fourth are the effects of heterogenous benchmarking.

Proof. To facilitate the derivation of instantaneous covariance between asset \( k \) and asset \( \ell \), \( \sigma_{kt}^{S_1} \sigma_{kt}^{S_2} \), the vector of asset exposures \( \sigma_{kt}^{S} \) can be written as
\[
\frac{\sigma_{kt}^{S}}{\sigma} = 1_k + \Omega_{kt}^1 \begin{pmatrix} \alpha_1 \\ 0 \\ 1 - \alpha_1 \end{pmatrix} + \Omega_{kt}^2 \begin{pmatrix} 0 \\ \alpha_2 \\ 1 - \alpha_2 \end{pmatrix},
\]
(A.26)
where $\Omega_{kt}^1$ and $\Omega_{kt}^2$ are characterized in (A.16) and (A.17), respectively. The covariation is derived by taking the cross product of two asset exposure vectors characterized by (15), (16) and (17). Specializing (A.16) and (A.17) to the different asset types we get

$$
\Omega_{kt}^1 = \frac{1}{C_3} (A_1 + A_2) w_{1t} f (1 - \alpha_1) + \frac{1}{C_3} w_{1t} w_{2t} (f (1 - \alpha_1) - f (1 - \alpha_2)), \quad (A.27)
$$

$$
\Omega_{kt}^1 = \frac{1}{C_1} (A_1 + A_2) w_{1t} f (\alpha_1) + \frac{1}{C_1} w_{1t} w_{2t} f (\alpha_1), \quad (A.28)
$$

$$
\Omega_{kt}^2 = -\frac{1}{C_2} w_{1t} w_{2t} f (\alpha_2), \quad (A.29)
$$

$$
\Omega_{kt}^2 = \frac{1}{C_3} (A_1 + A_2) w_{2t} f (1 - \alpha_2) - \frac{1}{C_3} w_{1t} w_{2t} (f (1 - \alpha_1) - f (1 - \alpha_2)), \quad (A.30)
$$

$$
\Omega_{kt}^2 = -\frac{1}{C_1} w_{1t} w_{2t} f (\alpha_1), \quad (A.31)
$$

$$
\Omega_{kt}^2 = \frac{1}{C_2} (A_1 + A_2) w_{2t} f (\alpha_2) + \frac{1}{C_2} w_{1t} w_{2t} f (\alpha_2). \quad (A.32)
$$

The result is obtained by simply plugging these results in the general formulation (A.15).

**Lemma A.2.** The following statements hold

$$
1 > \frac{\partial \hat{E}_1}{\partial \alpha_1} |_{\alpha_1=0} > 0,
$$

$$
\frac{\partial \hat{D}_1}{\partial \alpha_1} |_{\alpha_1=0} > 0,
$$

$$
\hat{E}_1 \geq \hat{E}_2 \text{ since } \alpha_1 = 0 \leq \alpha_2.
$$

**Proof.** Notice that $\alpha_1 = 0$ implies that $\hat{s}_{1t} = 0$, therefore, the only derivative that matters is

$$
\frac{\partial f (\alpha_1)}{\partial \alpha_1} |_{\alpha_1=0} = \left[ e^{\sigma^2 \alpha_1 (T-t)} \right] \sigma^2 (T-t) |_{\alpha_1=0} = \sigma^2 (T-t) \geq 0.
$$
For the first statement, by using this result we get

\[ 0 \leq \frac{\partial \hat{E}_1}{\partial \alpha_1} |_{\alpha_1=0} = \frac{w_{1t}w_{2t}}{C_1} |_{\alpha_1=0} \times \sigma^2 (T - t) \leq \frac{w_{1t}w_{2t}}{C_1} |_{\alpha_1=0} \leq 1. \]

The second to last inequality holds by assuming \( \sigma^2 (T - t) \leq 1. \)

For the second statement, we apply the same logic to

\[ \hat{D}_1 = \frac{(A_1 + A_2) f (\alpha_1) w_{1t}}{C_1}, \]

and show that the derivative is positive. For the last statement, the condition holds if and only if

\[ f (1 - \alpha_1) \geq f (1 - \alpha_2), \]

which is satisfied since \( 1 - \alpha_1 = 1 \geq 1 - \alpha_2. \)

\[ \square \]

**Lemma A.3.** If \( b_1 = b_2, I_{1t} = I_{2t}, \) and \( \alpha_1 \geq \alpha_2 \) then

- \( f(\alpha_1) w_{1t} \geq f(\alpha_2) w_{2t} \)
- \( f(\alpha_1) w_{2t} \geq f(\alpha_2) w_{1t} \)

**Proof.** Proving the first item, notice that when \( t = T, f(\alpha_1) = f(\alpha_2) = 0, \) and \( f(\alpha_1) w_{1t} = f(\alpha_2) w_{2t} = 0. \) Furthermore, for a given \( I_{it}, \) we observe that the derivative with respect to time is negative,

\[
\frac{\partial f(\alpha_i) w_{it}}{\partial t} = A_i b_i I_{it} e^{\left(\mu - \sigma^2 \alpha_i (1 - \alpha_i) \right) (T - t)} \left[ e^{\sigma^2 \alpha_i (T - t)} - 1 \right] \left( -\mu + \alpha_i (1 - \alpha_i) \sigma^2 \right) \\
+ A_i b_i I_{it} e^{\left(\mu - \sigma^2 \alpha_i (1 - \alpha_i) \right) (T - t)} \left[ e^{\sigma^2 \alpha_i (T - t)} - \alpha_i \right] \leq A_i b_i I_{it} e^{\left(\mu - \sigma^2 \alpha_i (1 - \alpha_i) \right) (T - t)} \left[ e^{\sigma^2 \alpha_i (T - t)} - \alpha_i \right] \alpha_i \sigma^2 [1 - \alpha_i - 1] \leq 0,
\]

where, \( i = 1, 2. \) Lastly, we observe that when \( t = 0, \)

\[
f(\alpha_1) w_{1t} = \frac{\lambda_1 e^{\mu T - \alpha_1 (1 - \alpha_1) \sigma^2 T}}{1 + b_1 I_0 e^{\mu T - \alpha_1 (1 - \alpha_1) \sigma^2 T}} b_1 I_{1t} f(\alpha_1) \geq \frac{\lambda_2 e^{\mu T - \alpha_2 (1 - \alpha_2) \sigma^2 T}}{1 + b_2 I_0 e^{\mu T - \alpha_2 (1 - \alpha_2) \sigma^2 T}} b_2 I_{2t} f(\alpha_2) = f(\alpha_2) w_{2t},
\]

because \( f(\alpha_1) \geq f(\alpha_2), \) \( \lambda_1 = \lambda_2, \) and

\[
\frac{e^{\mu T - \alpha_1 (1 - \alpha_1) \sigma^2 T}}{1 + b_1 I_0 e^{\mu T - \alpha_1 (1 - \alpha_1) \sigma^2 T}} \geq \frac{e^{\mu T - \alpha_2 (1 - \alpha_2) \sigma^2 T}}{1 + b_2 I_0 e^{\mu T - \alpha_2 (1 - \alpha_2) \sigma^2 T}}.
\]

We have shown that both derivatives with respect to time are negative, and in the two extreme cases, \( t = 0 \) and \( t = T, f(\alpha_1) w_{1t} \geq f(\alpha_2) w_{2t}, \) which implies that this relation holds for all intermediate time \( t \in [0, T]. \) Proving the second item notice that when \( t = T, f(\alpha_1) = f(\alpha_2) = 0, \) and \( f(\alpha_1) w_{2t} = \)
\[ f(\alpha_2) w_{1t} = 0. \] Furthermore, for a given \( I_{it} \), we observe that the derivative with respect to time is negative,

\[
\frac{\partial f(\alpha_1) w_{2t}}{\partial t} = A_2 b_2 I_{2t} e^{(\mu - \sigma^2 \alpha_2(1-\alpha_2))(T-t)} \left[ e^{\sigma^2 \alpha_1(T-t)} - 1 \right] (-\mu + \alpha_2 (1 - \alpha_2) \sigma^2) \\
+ A_2 b_2 I_{2t} e^{(\mu - \sigma^2 \alpha_2(1-\alpha_2))(T-t)} \left[ e^{\sigma^2 \alpha_1(T-t)} \right] (-\sigma^2 \alpha_1) \\
\leq A_2 b_2 I_{2t} e^{(\mu - \sigma^2 \alpha_2(1-\alpha_2))(T-t)} \left[ e^{\sigma^2 \alpha_1(T-t)} \right] \alpha_2 \sigma^2 \left[ 1 - \alpha_2 - 1 \right] \leq 0,
\]

where the last inequality follows from \( \alpha_1 \geq \alpha_2 \). Also,

\[
\frac{\partial f(\alpha_2) w_{1t}}{\partial t} = A_1 b_1 I_{1t} e^{(\mu - \sigma^2 \alpha_1(1-\alpha_1))(T-t)} \left[ e^{\sigma^2 \alpha_2(T-t)} - 1 \right] (-\mu + \alpha_1 (1 - \alpha_1) \sigma^2) \\
+ A_1 b_1 I_{1t} e^{(\mu - \sigma^2 \alpha_1(1-\alpha_1))(T-t)} \left[ e^{\sigma^2 \alpha_2(T-t)} \right] (-\sigma^2 \alpha_2) \\
\leq A_1 b_1 I_{1t} e^{(\mu - \sigma^2 \alpha_1(1-\alpha_1))(T-t)} \left[ e^{\sigma^2 \alpha_2(T-t)} \right] (-\alpha_1 \alpha_2) \sigma^2 \leq 0,
\]

where the last inequality follows from \( \alpha_1 \geq \alpha_2 \). Lastly, we observe that when \( t = 0 \), \( f(\alpha_2) w_{1t} \leq f(\alpha_1) w_{2t} \) if and only if

\[
\frac{e^{-\alpha_1(1-\alpha_1)\sigma^2 T}}{1 + b_1 I_0 e^{\mu T - \alpha_1(1-\alpha_1)\sigma^2 T}} \left[ e^{\sigma^2 \alpha_2 T} - 1 \right] \leq \frac{e^{-\alpha_2(1-\alpha_2)\sigma^2 T}}{1 + b_2 I_0 e^{\mu T - \alpha_2(1-\alpha_2)\sigma^2 T}} \left[ e^{\sigma^2 \alpha_1 T} - 1 \right].
\]

Rearranging the equation we get

\[
e^{-\alpha_1(1-\alpha_1)\sigma^2 T} \left( 1 + b_2 I_0 e^{\mu T - \alpha_2(1-\alpha_2)\sigma^2 T} \right) \left[ e^{\sigma^2 \alpha_2 T} - 1 \right] \leq e^{-\alpha_2(1-\alpha_2)\sigma^2 T} \left( 1 + b_1 I_0 e^{\mu T - \alpha_1(1-\alpha_1)\sigma^2 T} \right) \left[ e^{\sigma^2 \alpha_1 T} - 1 \right].
\]

Because \( \alpha_1 \geq \alpha_2 \), this relation holds if

\[
e^{(-\alpha_1(1-\alpha_1) + \alpha_2)\sigma^2 T} \leq e^{(-\alpha_2(1-\alpha_2) + \alpha_1)\sigma^2 T},
\]

which is true since \((-\alpha_1 (1 - \alpha_1) - \alpha_1)\) attains minimum at \( \alpha_1 = 1 \). We have shown that both derivatives with respect to time are negative, and in the two extreme cases, \( t = 0 \) and \( t = T \), \( f(\alpha_1) w_{2t} \geq f(\alpha_2) w_{1t} \), which implies that this relation holds for all intermediate time \( t \in [0, T] \).

**Proof of Proposition 4.** Illustratively, showing that \( \rho_S^S(dR_{1t}, dR_{3t}) \geq 0 \) is done by observing the matrix in (18). The sign of this correlation can be obtained by applying the dot product of the first and the third rows of the matrix. A careful look at the matrix reveals that both the first and third rows have identical signs. Therefore, the dot product results in a summation of only positive elements, which in turn results in a positive correlation. The correlation \( \rho_S^S(dR_{1t}, dR_{2t}) \) is formally

\[
\frac{\sigma_S^S}{\sigma^2} \frac{\sigma_S^2}{\sigma^2} = (\Omega_{2t} \alpha_1 + \Omega_{1t}^2 \alpha_2) + J_{12}^{\text{within type}} + J_{12}^{\text{across types}},
\]
as shown in Lemma A.1. Plugging the values from (A.29) and (A.31), we get that the correlation is negative if and only if

\[
\hat{E}_2 \alpha_1 + \hat{E}_1 \alpha_2 + \left( \hat{D}_1 + \hat{E}_1 \right) \hat{E}_2 \left[ (1 - \alpha_1)^2 + (\alpha_1)^2 \right] + \left( \hat{D}_2 + \hat{E}_2 \right) \hat{E}_1 \left[ (1 - \alpha_2)^2 + (\alpha_2)^2 \right] \\
\geq \left[ \left( \hat{D}_1 + \hat{E}_1 \right) \left( \hat{D}_2 + \hat{E}_2 \right) + \hat{E}_1 \hat{E}_2 \right] (1 - \alpha_1)(1 - \alpha_2).
\]

The idea is to decrease the left hand side and increase the right hand side. By increasing the right hand side we obtain

\[
\left[ \left( \hat{D}_1 + \hat{E}_1 \right) \left( \hat{D}_2 + \hat{E}_2 \right) + \hat{E}_1 \hat{E}_2 \right] (1 - \alpha_1)(1 - \alpha_2) \\
\leq \left[ \left( \hat{D}_1 + \hat{E}_1 \right) \left( \hat{D}_2 + \hat{E}_2 \right) + \hat{E}_1 \hat{E}_2 \right] ((1 - \alpha_1)^2 + (\alpha_2)^2) \\
\leq \left[ \left( \hat{D}_2 + \hat{E}_2 \right) \hat{E}_1 \right] (1 - \alpha_2)^2 + \left[ \left( \hat{D}_1 + \hat{E}_1 \right) \hat{E}_2 \right] (1 - \alpha_1)^2 + \hat{D}_1 \hat{D}_2 ((1 - \alpha_1)^2 + (1 - \alpha_2)^2)
\]

and by decreasing the left hand side we obtain

\[
\hat{E}_2 \alpha_1 + \hat{E}_1 \alpha_2 + \left( \hat{D}_1 + \hat{E}_1 \right) \hat{E}_2 \left[ (1 - \alpha_1)^2 + (\alpha_1)^2 \right] + \left( \hat{D}_2 + \hat{E}_2 \right) \hat{E}_1 \left[ (1 - \alpha_2)^2 + (\alpha_2)^2 \right] \\
\geq \hat{E}_2 \alpha_1 + \hat{E}_1 \alpha_2 + \left[ \left( \hat{D}_1 + \hat{E}_1 \right) \hat{E}_2 \right] (1 - \alpha_1)^2 + \left[ \left( \hat{D}_2 + \hat{E}_2 \right) \hat{E}_1 \right] (1 - \alpha_2)^2.
\]

Therefore, sufficient condition for \( \rho^*_i(dR_{1t}, dR_{2t}) \leq 0 \) is

\[
\hat{E}_2 \alpha_1 + \hat{E}_1 \alpha_2 \geq \hat{D}_1 \hat{D}_2 ((1 - \alpha_1)^2 + (1 - \alpha_2)^2).
\]

Plugging \( \hat{E}_2, \hat{E}_1, \hat{D}_1, \hat{D}_2, (A.28), (A.32), \) and \( C_{it}, (A.3), \) we obtain

\[
\frac{w_{1t}w_{2t}f(\alpha_2)}{c_t (c_t + w_{2t}f(\alpha_2))} \alpha_1 + \frac{w_{1t}w_{2t}f(\alpha_1)}{c_t (c_t + w_{1t}f(\alpha_1))} \alpha_2 \geq \frac{(A_1 + A_2)^2 f(\alpha_1) f(\alpha_2) w_{1t}w_{2t}}{c_t^2 (c_t + w_{1t}f(\alpha_1))(c_t + w_{2t}f(\alpha_2))} ((1 - \alpha_1)^2 + (1 - \alpha_2)^2).
\]

Simplifying this inequality we obtain

\[
f(\alpha_2) (c_t + w_{1t}f(\alpha_1)) \alpha_1 + f(\alpha_1) (c_t + w_{2t}f(\alpha_2)) \alpha_2 \geq \frac{(A_1 + A_2)^2 f(\alpha_1) f(\alpha_2)}{c_t} ((1 - \alpha_1)^2 + (1 - \alpha_2)^2).
\]

Taking the limits where \( I_{1t} \to 0 \) and \( I_{2t} \to 0 \) we get that \( w_{1t} \) and \( w_{2t} \) goes to zero as well. By doing so we increase the right hand side and decrease the left hand side.

\[
f(\alpha_2) \alpha_1 + f(\alpha_1) \alpha_2 \geq 2 f(\alpha_2) f(\alpha_1) \geq f(\alpha_1) f(\alpha_2) ((1 - \alpha_1)^2 + (1 - \alpha_2)^2),
\]

30
which is satisfied if $\sigma \leq \frac{\log 2}{T - t}$, because it guarantees that $f(\alpha_i) \leq \alpha_i$ for $\alpha_i \in [0, 1]$, $i = 1, 2$.

For the last condition, we get that $\rho_i^S(dR_{2t}, dR_{3t}) \leq 0$ if and only if

$$(\Omega_{3t}^2 \alpha_2 + \Omega_{2t}^1 (1 - \alpha_1) + \Omega_{2t}^2 (1 - \alpha_2)) + (\Omega_{2t}^1 \Omega_{3t}^1 [(1 - \alpha_1)^2 + (\alpha_1)^2] + \Omega_{2t}^2 \Omega_{3t}^2 [(1 - \alpha_2)^2 + (\alpha_2)^2]) + (\Omega_{2t}^1 \Omega_{3t}^2 + \Omega_{3t}^2 \Omega_{2t}^2) (1 - \alpha_1) (1 - \alpha_2) \leq 0.$$ 

Plugging the coefficients, (A.27), (A.30), (A.31) and (A.32), and rearranging we get

$$D_2 \alpha_2 + \left[ \hat{D}_2 + \hat{E}_2 \right] (1 - \alpha_2) + \left[ \hat{D}_2 + \hat{E}_2 \right] D_2 [(1 - \alpha_2)^2 + \alpha_2^2] + \left[ (\hat{D}_2 + \hat{E}_2) (D_1 + E_1 - E_2) + (E_1 - E_2) \hat{E}_2 \right] (1 - \alpha_1) (1 - \alpha_2) \leq \hat{E}_2 (1 - \alpha_1) + (E_1 - E_2) \alpha_2 + \left[ \hat{D}_2 + \hat{E}_2 \right] (E_1 - E_2) [(1 - \alpha_2)^2 + \alpha_2^2] + \hat{E}_2 (D_1 + E_1 - E_2) [(1 - \alpha_1)^2 + \alpha_1^2] + D_2 \hat{E}_2 (1 - \alpha_1) (1 - \alpha_2).$$

For a given $\alpha_1$, when $\alpha_2 = 0$ both $\hat{D}_2 = 0, \hat{E}_2 = 0$ and the correlation $\rho_i^S(dR_{2t}, dR_{3t}) = 0$. In contrast, for a given $\alpha_1$, when $\alpha_2 = 1$ both $D_2 = 0, E_2 = 0$, and the inequality simplifies to

$$0 \leq \hat{E}_1 (1 - \alpha_2) + E_1 \left( 1 + \hat{D}_2 + \hat{E}_2 \right) + \hat{E}_2 (D_1 + E_1) [(1 - \alpha_1)^2 + \alpha_1^2],$$

which is always satisfied. To complete the proof we show that the derivative $\frac{\partial \rho_i^S(dR_{2t}, dR_{3t})}{\partial \alpha_2} |_{\alpha_2 = 0} > 0$. By taking the derivative and plugging $\hat{D}_2 = 0, \hat{E}_2 = 0$ we obtain

$$\frac{\partial \rho_i^S(dR_{2t}, dR_{3t})}{\partial \alpha_2} |_{\alpha_2 = 0} > 0 \iff D_2 + \frac{\partial \hat{D}_2 + \partial \hat{E}_2}{\partial \alpha_2} (1 + D_2) + \left[ \frac{\partial \hat{D}_2 + \partial \hat{E}_2}{\partial \alpha_2} (D_1 + E_1 - E_2) + \frac{\partial \hat{E}_2}{\partial \alpha_2} (E_1 - E_2) \right] (1 - \alpha_1) + \frac{\partial \hat{E}_2}{\partial \alpha_2} (D_1 + E_1 - E_2) [(1 - \alpha_1)^2 + \alpha_1^2] + \frac{\partial \hat{E}_2}{\partial \alpha_2} D_2 (1 - \alpha_1).$$

In order to show that this inequality holds, we break it to few segments. By the statements in Lemma A.2 we have

$$1 > \frac{\partial \hat{E}_2}{\partial \alpha_2} |_{\alpha_2 = 0} > 0,$$

$$\frac{\partial \hat{D}_2}{\partial \alpha_2} |_{\alpha_2 = 0} > 0,$$

$$E_2 \geq E_1, \quad \alpha_2 = 0 \leq \alpha_1.$$
Second, because $D_1 < 1$ we have
\[ \alpha_1 + D_1 (1 - \alpha_1) > D_1 \geq D_1 [(1 - \alpha_1)^2 + \alpha_1^2], \]
which implies that
\[ \frac{\partial \hat{E}_2}{\partial \alpha_2} (\alpha_1 + D_1 (1 - \alpha_1)) > \frac{\partial \hat{E}_2}{\partial \alpha_2} (D_1 [(1 - \alpha_1)^2 + \alpha_1^2]) . \]

By using this relation we simplify the inequality to
\[ D_2 + \frac{\partial \hat{D}_2 + \partial \hat{E}_2}{\partial \alpha_2} D_2 + \frac{\partial \hat{D}_2 + \partial \hat{E}_2}{\partial \alpha_2} \left[ (E_1 - E_2) + \frac{\partial \hat{E}_2}{\partial \alpha_2} (E_1 - E_2) \right] (1 - \alpha_1) \]
\[ > (E_1 - E_2) + \frac{\partial \hat{D}_2 + \partial \hat{E}_2}{\partial \alpha_2} (E_1 - E_2) \left[ (1 - \alpha_1)^2 + \alpha_1^2 \right] + \frac{\partial \hat{E}_2}{\partial \alpha_2} D_2 (1 - \alpha_1) . \]

Third, by using
\[ \frac{\partial \hat{E}_2}{\partial \alpha_2} D_2 (1 - \alpha_1) \leq \frac{\partial \hat{E}_2}{\partial \alpha_2} D_2, \]
\[ \frac{\partial \hat{D}_2 + \partial \hat{E}_2}{\partial \alpha_2} (E_1 - E_2) (1 - \alpha_1) \geq \frac{\partial \hat{D}_2 + \partial \hat{E}_2}{\partial \alpha_2} (E_1 - E_2), \]
we obtain
\[ D_2 + \frac{\partial \hat{D}_2 + \partial \hat{E}_2}{\partial \alpha_2} (1 + D_2) + \frac{\partial \hat{E}_2}{\partial \alpha_2} (E_1 - E_2) (1 - \alpha_1) \geq (E_1 - E_2) + \frac{\partial \hat{E}_2}{\partial \alpha_2} (E_1 - E_2) \left[ (1 - \alpha_1)^2 + \alpha_1^2 \right] . \]

This relation holds because $\frac{\partial \hat{E}_2}{\partial \alpha_2} < 1$, and $E_1 \leq E_2$, as stated in Lemma A.2. Summarizing, we have shown that when $\alpha_2 = 0$, $\rho^S_i (dR_{2t}, dR_{3t}) = 0$, when $\alpha_1 = 1$, $\rho^S_i (dR_{2t}, dR_{3t}) < 0$, and $\frac{\partial \rho^S_i (dR_{2t}, dR_{3t})}{\partial \alpha_2} |_{\alpha_2 = 0} > 0$. Therefore, by continuity, there are $\alpha_2^*$ and $\alpha_1^*$ as the theory suggests.

\[ \square \]

**Proof of Proposition 5.** By using (A.20) and (A.25) and plugging $\alpha_2 = 1$ we obtain
\[ \frac{\sigma_3^S \sigma_3^S}{\sigma^2} = 2 \Omega_{3t}^2 (1 - \alpha_1) + \left( \Omega_{3t}^1 \Omega_{3t}^1 \left[ (1 - \alpha_1)^2 + (\alpha_1)^2 \right] + \Omega_{3t}^2 \Omega_{3t}^2 \right), \]
\[ \frac{\sigma_2^S \sigma_2^S}{\sigma^2} = 2 \Omega_{2t}^2 + \Omega_{2t}^1 \Omega_{2t}^1 \left[ (1 - \alpha_1)^2 + (\alpha_1)^2 \right] + \Omega_{2t}^2 \Omega_{2t}^2. \]
Furthermore, observing that $E_2 = D_2 = 0$ when $\alpha_2 = 1$, and plugging the coefficients (A.29), (A.27), (A.32), and (A.30) we get

$$
\frac{\sigma_3^S \sigma_3^S}{\alpha^2} = 2(D_1 + E_1)(1 - \alpha_1) + \left((E_1)^2 + (D_1 + E_1)^2 \left[(1 - \alpha_1)^2 + (\alpha_1)^2\right]\right),
$$

$$
\frac{\sigma_2^S \sigma_2^S}{\alpha^2} = 2\left(\hat{D}_2 + \hat{E}_2\right) + \left(\hat{D}_2 + \hat{E}_2\right)^2 + \hat{E}_2^2 \left[(1 - \alpha_1)^2 + (\alpha_1)^2\right].
$$

By assuming that $I_{1t} = I_{2t}$, and $b_1 = b_2$, Lemma (A.3) item (1), shows that $f(\alpha_2) w_{2t} \geq f(1 - \alpha_1) w_{1t}$, which then implies that

$$
\hat{D}_2 = \frac{(A_1 + A_2) f(\alpha_2) w_{2t}}{C_2} = \frac{(A_1 + A_2) f(\alpha_2) w_{2t}}{c_1(c_t + w_{2t} f(\alpha_2))} \geq \frac{(A_1 + A_2) f(1 - \alpha_1) w_{1t}}{c_1(c_t + w_{1t} f(1 - \alpha_1))} = \frac{(A_1 + A_2) f(1 - \alpha_1) w_{1t}}{C_3} = D_1. \quad (A.33)
$$

Furthermore, Lemma (A.3) item (2) shows that $f(\alpha_2) w_{1t} \geq f(1 - \alpha_1) w_{2t}$. Using this fact we get the relation

$$
f(\alpha_2) f(1 - \alpha_1) w_{2t} - f(\alpha_2) f(1 - \alpha_1) w_{1t} \leq f(\alpha_2) f(1 - \alpha_1) w_{2t} - f(1 - \alpha_1) f(1 - \alpha_1) w_{2t} = f(1 - \alpha_1) w_{2t} (f(\alpha_2) - f(1 - \alpha_1)).
$$

Because $\sigma^2 \leq \frac{\log 2}{T-1}$ implies that $f(1 - \alpha_1) \leq 1$, and we get

$$
f(1 - \alpha_1) w_{2t} (f(\alpha_2) - f(1 - \alpha_1)) \leq w_{2t} (f(\alpha_2) - f(1 - \alpha_1)) \leq c_t (f(\alpha_2) - f(1 - \alpha_1)),$$

which implies that

$$
c_t (f(\alpha_2) - f(1 - \alpha_1)) \geq f(\alpha_2) f(1 - \alpha_1) w_{2t} - f(\alpha_2) f(1 - \alpha_1) w_{1t}.
$$

Rearranging the equation we get

$$
f(\alpha_2) (c_t + f(1 - \alpha_1) w_{1t}) \geq f(1 - \alpha_1) (c_t + f(\alpha_2) w_{2t}),
$$

which implies that

$$
\hat{E}_2 = \frac{w_{1t} w_{2t} f(\alpha_2)}{C_2} = \frac{w_{1t} w_{2t} f(\alpha_2)}{c_t(c_t + w_{2t} f(\alpha_2))} \geq \frac{w_{1t} w_{2t} f(1 - \alpha_1)}{c_t(c_t + w_{1t} f(1 - \alpha_1))} = \frac{w_{1t} w_{2t} f(1 - \alpha_1)}{C_3} = E_1. \quad (A.34)
$$
Let \( p \equiv [(1 - \alpha_1)^2 + (\alpha_1)^2] \), the inequalities in (A.33) and (A.34) implies that
\[
\frac{\sigma_3^S \sigma_3^S}{\sigma^2} \leq \frac{\sigma_3^S \sigma_3^S}{\sigma^2} + 2 (D_1 + E_1) \alpha_1 + (D_1 + E_1)^2 (1 - p) + \left( \hat{E}_2 \right)^2 p - (E_1)^2 \\
= 2 (D_1 + E_1) + (D_1 + E_1)^2 + \left( \hat{E}_2 \right)^2 p \\
\leq 2 (\hat{D}_2 + \hat{E}_2) + (\hat{D}_2 + \hat{E}_2)^2 + \left( \hat{E}_2 \right)^2 p = \frac{\sigma_2^S \sigma_2^S}{\sigma^2},
\]
where the first inequality holds because \( \hat{E}_1 \geq E_2 \) and
\[
(E_1)^2 \leq (E_1)^2 (1 - p) + \left( \hat{E}_2 \right)^2 p \leq 2 (D_1 + E_1) (1 - \alpha_1) + (D_1 + E_1)^2 (1 - p) + \left( \hat{E}_2 \right)^2 p.
\]
Therefore, the inequality
\[
\frac{\sigma_3^S \sigma_3^S}{\sigma^2} \leq \frac{\sigma_3^S \sigma_3^S}{\sigma^2} + 2 (D_2 + E_2) (1 - \alpha_2) + (D_2 + E_2)^2 (1 - p) + E_1^2 p - E_2^2
\]
holds, and we obtain \( \frac{\sigma_3^S \sigma_3^S}{\sigma^2} \leq \frac{\sigma_3^S \sigma_3^S}{\sigma^2} \) when \( \alpha_2 = 1 \). Now, let us suppose that \( \alpha_2 = 0 \). Simplifying the volatility terms we obtain
\[
\frac{\sigma_3^S \sigma_3^S}{\sigma^2} = 2 (\Omega_{3t}^1 (1 - \alpha_1) + \Omega_{3t}^2) + (\Omega_{3t}^1 \Omega_{3t}^1 [-(1 - \alpha_1)^2 + (\alpha_1)^2] + \Omega_{3t}^2 \Omega_{3t}^2) + 2 \Omega_{3t}^1 \Omega_{3t}^2 (1 - \alpha_1),
\]
\[
\frac{\sigma_2^S \sigma_2^S}{\sigma^2} = (\Omega_{2t}^1 \Omega_{2t}^1 [(1 - \alpha_1)^2 + (\alpha_1)^2] + \Omega_{2t}^2 \Omega_{2t}^2) + 2 \Omega_{2t}^1 \Omega_{2t}^2 (1 - \alpha_1).
\]
Furthermore, \( \Omega_{2t}^1 = \Omega_{2t}^2 = 0 \) when \( \alpha_2 = 0 \). Therefore,
\[
\frac{\sigma_3^S \sigma_3^S}{\sigma^2} \geq \frac{\sigma_2^S \sigma_2^S}{\sigma^2} = 0.
\]
Because \( \sigma_3^S \sigma_3^S \) is the norm of the vector it is always greater or equal to zero and the inequality holds. Summarizing, we have shown \( \alpha_2 = 1 \) implies that \( \frac{\sigma_3^S \sigma_3^S}{\sigma^2} \geq \frac{\sigma_2^S \sigma_2^S}{\sigma^2} \), and \( \alpha_2 = 0 \) implies that \( \frac{\sigma_3^S \sigma_3^S}{\sigma^2} \leq \frac{\sigma_2^S \sigma_2^S}{\sigma^2} \). Therefore, there exist \( \sigma^2 \) and \( \alpha^2 \) as the theory suggests. \( \square \)
References


