BCF mini course: Deep Learning and Macro-Finance Models

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Roadmap

- Part-1: Introduction to numerical methods, challenges faced by traditional methods
  - Why neural networks and deep learning
  - Function approximators
  - Comparison with existing methods
- Part-2: Deep learning principles, high-dimensional optimization techniques in machine learning
  - Gradient descent and variants
  - Under the hood: Activation functions, Parameter initialization
  - Object oriented programming principles
- Part-3: Application to solve macro-finance models with aggregate shocks

References

- Goutham Gopalakrishna. ALIENs and Continuous Time Economies. 2021. SSRN Working paper.
- Justin Sirignano and Konstantinos Spiliopoulos. DGM: A deep learning algorithm for solving partial differential equations. 2018a. Journal of Computational Physics.
- Victor Duarte. Machine Learning for Continuous-Time Economics. 2017. SSRN Working paper.
- Jesús Fernández-Villaverde, Samuel Hurtado, and Galo Nuno. Financial Frictions and the Wealth Distribution. 2022. Econometrica (forthcoming).

- Course materials (slides and code): Github page.

ALIENs: What is it about?

- **ENs**: Use neural network to solve general equilibrium continuous time finance models to capture global dynamics (portfolio choice, macro-finance, monetary policy)

1. Portfolio Choice: Merton (1971), Cochrane et al (2008), Martin (2013)
2. Macro-Finance: He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014), Di Tella (2017)
3. Monetary Theory: Silva (2020), Brunnermeier and Sannikov (2016)

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ALIENs: What is it about?

- **AL**: Actively learn about state space with stark non-linearity/large prediction error
- **I**: Encode economic information as regularizer
- **ENs**: Use neural network to solve general equilibrium continuous time finance models to capture global dynamics (portfolio choice, macro-finance, monetary policy)

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General setup

- $$U_t = E_t\Big[\int_t^{\infty} f(c_s, U_s)ds\Big] \tag{1}$$

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- Exogenous dividend process of risky asset

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- There is also a risk free debt market (pays return r). Risky asset has price of risk ζ_t, and volatility σ_t^R
- Problem of the agent is

$$\sup_{\hat{c},\theta} \quad U_t \tag{3}$$

$$\text{s.t} \quad \frac{dw_t}{w_t} = (r + \underbrace{\theta_t}_{\text{port. choice}} \underbrace{\zeta_t}_{\text{price of risk}} - \hat{c}_t)dt + \theta_t \underbrace{\sigma_t^R}_{\text{ret. volatility}} dZ_t \tag{4}$$

- If g, σ, r are time varying, then we have a multi-dimensional problem

HJB

- HJB is

$$\sup_{\hat{c}_t,\theta_t} f(c_t, U_t) + E_t(dU_t) = 0$$

- Conjecturing $U = \frac{Jw^{1-\gamma}}{1-\gamma}$, where J is the stochastic opportunity process and γ is the risk aversion, the HJB equation reduces to

$$\mu^J(\boldsymbol{x}, J)J = \sum_{i=1}^{d} \mu^{x_i}(\boldsymbol{x}, J)\frac{\partial J}{\partial x_i} + \sum_{i,j=1}^{d} b^{i,j}(\boldsymbol{x}, J)\frac{\partial^2 J}{\partial x_i \partial x_j} \tag{5}$$

  1. State variables are x. Could be high-dimensional (large d)
  2. μ^J, μ^x, and $b^{i,j}$ are linear, advection, and diffusion coefficients

- PDE (5) can be highly non-linear elliptical PDE depending on the problem
- Past literature: Convert it into **quasi-linear parabolic PDE** and use finite difference → slowly introduce non-linearity through

$$\mu^J(\boldsymbol{x}, J^{old})J = \frac{\partial J}{\partial t} + \sum_{i=1}^{d} \mu^{x_i}(\boldsymbol{x}, J^{old})\frac{\partial J}{\partial x_i} + \sum_{i,j=1}^{d} b^{i,j}(\boldsymbol{x}, J^{old})\frac{\partial^2 J}{\partial x_i \partial x_j} \tag{6}$$

- Works well in low dimensions, but breaks down in high dimensions (d'Adrien and Vandeweyer, 2019)

Methodology overview

- Focus of this part is to introduce a technique to solve macro models involving PDEs of type (5) in high dimensions
  1. Benchmark model (BS2016 with recursive preference)
  2. Capital misallocation model with productivity shock (Gopalakrishna 2021)

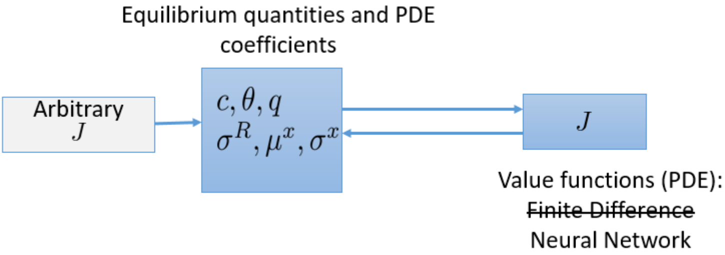

Figure: Overview of methodology.

Neural network solution method

$$f := \frac{\partial \hat{J}}{\partial t} + \sum_{i}^{d} \mu^{i}(\boldsymbol{x}) \frac{\partial \hat{J}}{\partial x_i} + \sum_{i,j=1}^{d} b^{i,j}(\boldsymbol{x}) \frac{\partial^2 \hat{J}}{\partial x_i \partial x_j} - \mu^{J} \hat{J} = 0;$$

$$\forall (t, \boldsymbol{x}) \in [T - k\Delta t, T - (k-1)\Delta t] \times \Omega$$

$$\hat{J} = \tilde{J}_0 \quad \forall (t, \boldsymbol{x}) \in (T - (k-1)\Delta t) \times \Omega;$$

where $\hat{J}$ is a neural network object with parameters Θ, and f is the PDE residual.

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where $\hat{J}$ is a neural network object with parameters Θ, and f is the PDE residual. Can be seen as a classical constrained optimization problem

Optimization

$$\Theta^* = \underset{\Theta}{argmin} \quad \hat{J} - \tilde{J}_0$$
$$\text{s.t.} \quad f = 0$$

Neural network solution method

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Optimization

$$\Theta^* = \underset{\Theta}{argmin} \quad \hat{J} - \tilde{J}_0$$

$$\text{s.t.} \quad \int_t \int_{\boldsymbol{x}} |f|^2 dt d\boldsymbol{x} = 0$$

Neural network solution method

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$$\hat{J} = \tilde{J}_0 \quad \forall (t, \boldsymbol{x}) \in (T - (k-1)\Delta t) \times \Omega;$$

$$\frac{\partial \hat{J}}{\partial x} = J_0 \quad \forall (t, \boldsymbol{x}) \in (T - (k-1)\Delta t) \times \partial\Omega;$$

- Mesh free since we can randomly sample from the state space $(t, \boldsymbol{x})$ to train the neural network

Neural network solution method

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- Mesh free since we can randomly sample from the state space $(t, \boldsymbol{x})$ to train the neural network
- Sparse training points in region of importance leads to instability in future iterations. Solution: Track subdomain Ω_c and sample more points from there

$$f = 0 \quad \forall (t, \boldsymbol{x}) \in [T - k\Delta t, T - (k-1)\Delta t] \times \Omega_c;$$
$$\hat{J} = \tilde{J}_0 \quad \forall (\boldsymbol{x}, t) \in (T - (k-1)\Delta t) \times \Omega_c;$$

- The subdomain Ω_c is found by inspecting the PDE coefficients which are determined using previous value $\tilde{J}$

Neural network solution method

$$f := \frac{\partial \hat{J}(\mathbf{x}|\Theta)}{\partial t} + \sum_{i}^{d} \mu^{i}(\mathbf{x}) \frac{\partial \hat{J}(\mathbf{x}|\Theta)}{\partial x_i} + \sum_{i,j=1}^{d} b^{i,j}(\mathbf{x}) \frac{\partial^2 \hat{J}(\mathbf{x}|\Theta)}{\partial x_i \partial x_j}$$

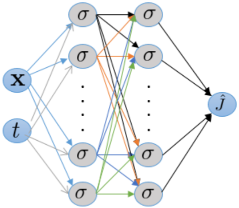
$$- \mu^{J} \hat{J}(\mathbf{x}|\Theta) = 0; \quad \forall (t, \mathbf{x}) \in [T - k\Delta t, T - (k-1)\Delta t] \times \Omega$$

$$\hat{J}(\mathbf{x}|\Theta) = \tilde{J}_0; \quad \forall (t, \mathbf{x}) \in (T - (k-1)\Delta t) \times \Omega;$$

$$(f = 0; \quad \forall (t, \mathbf{x}) \in [T - k\Delta t, T - (k-1)\Delta t] x \Omega_c;$$

$$\hat{J}(\mathbf{x}|\Theta) = \tilde{J}_0; \quad \forall (t, \mathbf{x}) \in (T - (k-1)\Delta t)\Omega_c); \rightarrow \text{Active learning}$$

$$\frac{\partial \hat{J}(\mathbf{x}|\Theta)}{\partial \mathbf{x}} = J_0; \quad \forall (t, \mathbf{x}) \in (T - (k-1)\Delta t) \times \partial\Omega;$$



$\mathbf{X} \in \mathbb{R}^d$ Space dimension

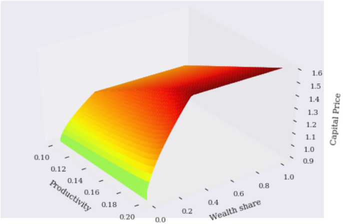
$t \in [0, T]$ Time dimension

σ Tanh activation function. $\sigma(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\hat{J}(x \mid \Theta)$ Output from neural network

Active learning

Example from Gopalakrishna (2021): Macro-finance model with 2 state variables (productivity, wealth share)



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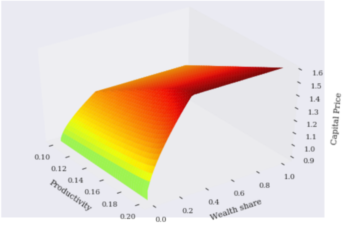

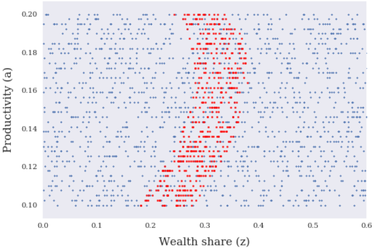

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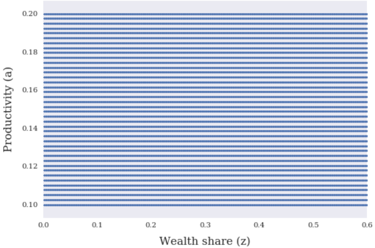
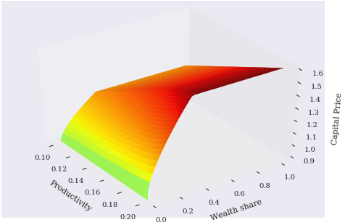
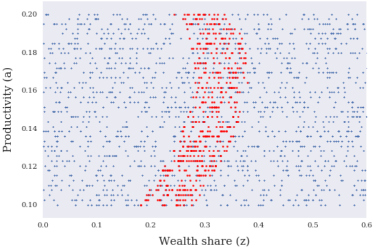
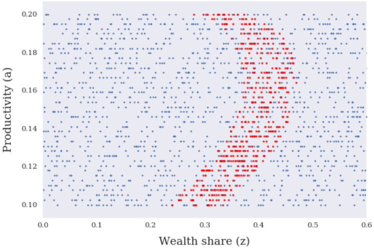








Solution technique: ALIENs

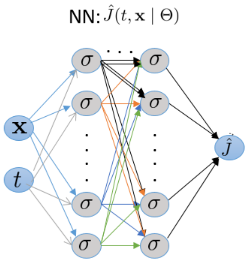
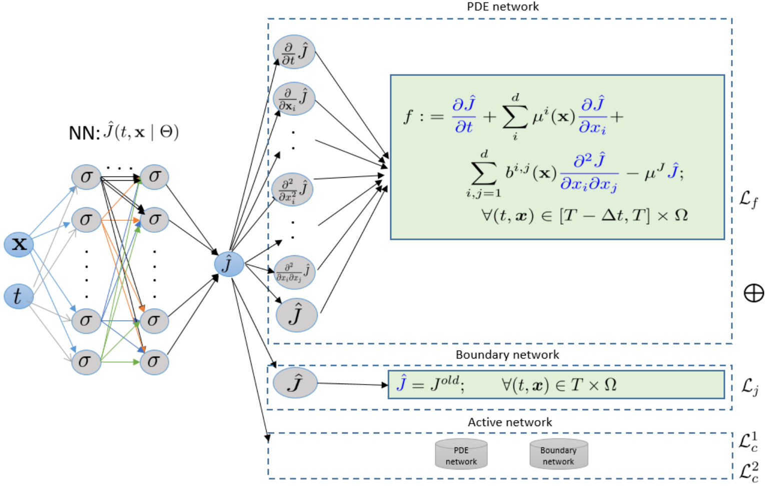


Figure: Methodology.

Solution technique: ALIENs



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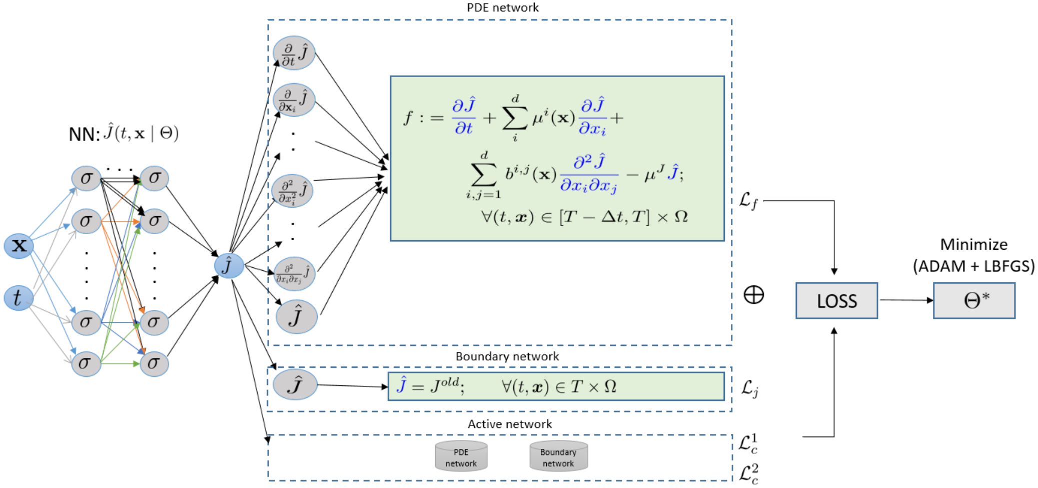


Figure: Methodology.

ALIENs

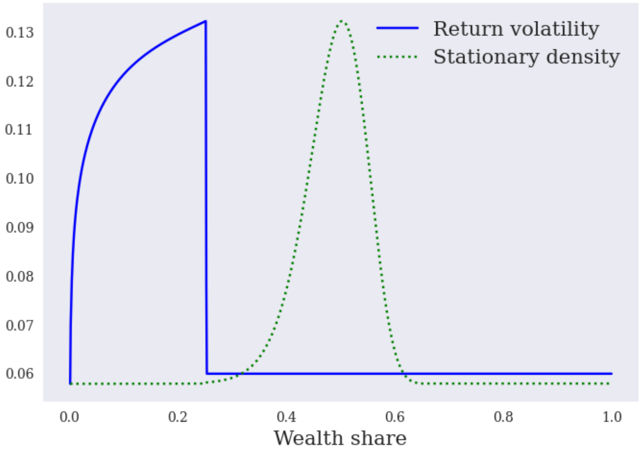
$$\mathcal{L} = \lambda_f \mathcal{L}_f + \lambda_j \mathcal{L}_j + \lambda_b \mathcal{L}_b + \lambda_c^1 \mathcal{L}_c^1 + \lambda_c^2 \mathcal{L}_c^2 \tag{7}$$

where

$$\begin{aligned}
&\text{PDE loss} && \mathcal{L}_f = \frac{1}{N_f} \sum_{i=1}^{N_f} |f(\mathbf{x}_f^i, t_f^i)|^2 \\
&\text{Bounding loss-1} && \mathcal{L}^j = \frac{1}{N_j} \sum_{i=1}^{N_j} |\hat{J}(\mathbf{x}_j^i, t_j^i) - \tilde{J}_0^i|^2 \\
&\text{Active loss-1} && \mathcal{L}_c^2 = \frac{1}{N_c} \sum_{i=1}^{N_c} |f(\mathbf{x}_c^i, t_c^i)|^2 \\
&\text{Active loss-2} && \mathcal{L}_c^1 = \frac{1}{N_c} \sum_{i=1}^{N_c} |\hat{J}(\mathbf{x}_c^i, t_c^i) - \tilde{J}_0^i|^2
\end{aligned}$$

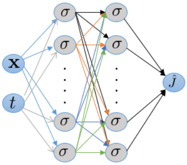
Active Learning vs Simulation method

- ALIENs actively learn the region of sharp transition and samples more points → faster convergence
- Sampling procedure is complementary to simulation based methods (Azinovic et al (2018), Villaverde et al (2020)), but also works for models with rare events and financial constraints that bind far away from the steady state



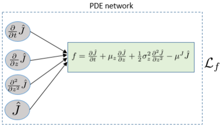
Automatic differentiation in practice

Approximating J using a neural network




```
def J(z,t):
        J = neural_net(tf.concat([z,t],1),weights,biases)
        return J
```

Constructing regularizer: 1D model




```
def f(z,t):
        J = J(z,t)
        J_t = tf.gradients(J,t)[0]
        J_z = tf.gradients(J,z)[0]
        J_zz = tf.gradients(J_z,z)[0]
        f = J_t + advection * J_z + diffusion * J_zz - linearTerm * J
        return f
```

Automatic differentiation in practice

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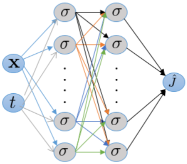


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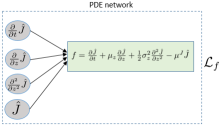


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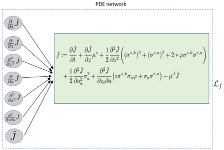


```
def f(z,a,t):
        J = J(z,a,t)
        J_t = tf.gradients(J,t)[0]
        J_z = tf.gradients(J,z)[0]
        J_a = tf.gradients(J,a)[0]
        J_zz = tf.gradients(J_z,z)[0]
        J_aa = tf.gradients(J_a,a)[0]
        J_az = tf.gradients(J_a,z)[0]
        f = J_t + advection_z * J_z + advection_a * J_a + diffusion_z * J_zz +
                        diffusion_a * J_aa  + crossTerm * J_az - linearTerm * J
        return f
```

Horovod

- Data parallelism as opposed to Model parallelism
- Horovod uses ringAllReduce operation to average gradients (improves efficiency)

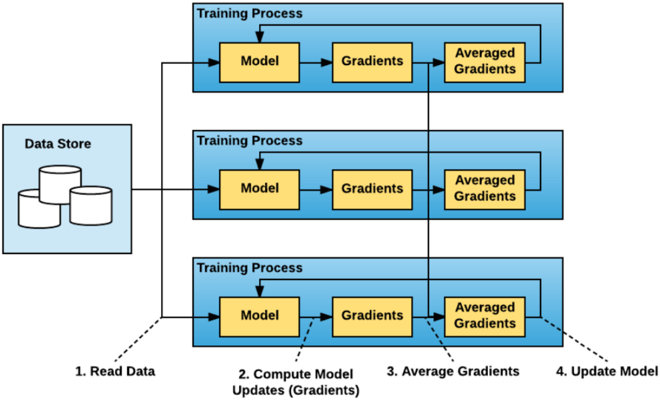

Figure: Source: https://eng.uber.com/horovod/

Horovod

```
def J():
        ...
def f():
        ...

hvd.init() #initialize Horovod
config = tf.ConfigProto() #pin GPUs to processes
config.gpu_options.visible_device_list = str(hvd.local_rank()) #assign chief worker
config.gpu_options.allow_growth = True #enable GPU
sess= tf.Session(config=config) #Configure tensorflow
if hvd.rank()==0:
        ... #assign a piece of data to chief worker
else:
        while hvd.rank() < hvd.size():
                ... #assign a piece of data to each worker

def build_model():
        #initialize parameters using Xavier initialization
        #parametrize the function J using J()
        #buld loss function using net_f()
        #set up tensorflow optimizer in the variable name opt
        optimizer = hvd.DistributedOptimizer(opt)
        #minimize loss
        #initialize Tensorflow session
        bcast = hvd.broadcast_global_variables(0) #Broadcast parameters to all workers
        sess.run(bcast)
        #train the deep learning model
```

HPC

Interactive mode

Sinteract -q gpu -p gpu -g gpu -m 12G -t 10:00:00
virtualenv –system-site-packages venv-for-tf
source ./venv-for-tf/bin/activate
pip install –user –no-cache-dir tensorflow-gpu==2.7.0

ipythonCores: 1
Tasks: 1
Time: 10:00:00
Memory: 128G
Partition: gpu
Account: sfi-pcd
Jobname: interact
Resource: gpu
QOS: gpu
salloc: job 124415 allocated